Part III — Infinite Games

Lecture Notes

Lent 2020

Based on lectures by Prof Benedikt Löwe. The scribe is H. T. Chau, who uses a style file by Evan Chen available on GitHub. Disclaimer: These notes do not necessarily reflect what is in the lectures, and contain non-standard shorthands. The official schedule is available on the lecturer's page, and archived on 22 Mar 2020.

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§0 Introduction

Question 0.1. What do we mean by an *infinite game*?

Economists think this means a game where players have infinite choices. Some thinks it means an infinite sequence of finite games.

Definition 0.2 (Infinite games)**.** An **infinite game** is one that has infinite length, two players, zero sum, perfect information, and perfect recall.

Definition 0.3. A game has **infinite length** if plays are infinite sequences of moves. (So any finite segment does not necessarily tell useful properties.)

We require **two players** because there are three player games in which no individual can force a win but only coalitions can, e.g. one in which III decides whether I or II wins. (Henceforth I is almost never a pronoun.)

Zero sum usually means $x + y = 0$ for all utility entries (x, y), but here we mean something stronger: we only look at *win-lose* games, i.e. the utilities are $(+1, -1)$ or $(-1, +1)$.

Example 0.4 (Non-zero sum)

Stag hunt

	II	
I	2,2	0,1
	1,0	1,1

Prisoners' dilemma

	II	
I	2,2	0,1
	1,0	1,1

Battle of the sexes

	II	
I	3,2	1,1
	0,0	2,3

	II	
I	1,1	0,0
	0,0	1,1

These require coordination and cooperation.

We look at win-lose games so we can think of the payoff as set

$$A = \{x : \text{player I gets 1 if the game has play } x\}.$$

Definition 0.5 (Perfect info)**.** At any point in the game, players know everything that happens.

Example 0.6 (Non-perfect info)

In a card game, I can make the move (i.e. transition between game states)

$$(\text{Hand}_1 = \clubsuit 2 \clubsuit 3,\ \text{Hand}_2 = \heartsuit 2 \heartsuit 3) \mapsto (\clubsuit 2 \text{ in hand} + \clubsuit 3 \text{ played},\ \text{Hand}_2)$$

but I do not know Hand_2 so I do not know which move I has made.

Definition 0.7 (perfect recall)**.** Players recall everything that happened, with unbounded memory.

Both perfect info and perfect recall will be encoded in our notion of strategy.

Fact 0.8 (history)**.** *Application of set theory to the theory of chess* (Zermelo 1913) uses naïve set theory to show that either black or white has an at-least-drawing strategy.

In Poland Lwów (now Lviv in Ukraine) 1930/40s (this period has large influence including Polish space in topology) was Scottish Café, where there was a notebook *Scottish Book*. Banach, Mazur, Ulam et al wrote problems, solutions, and ideas in it, and the notebook was only published in 1957. Some ideas includes

$$\text{Inf Games} \longleftrightarrow \text{Topology} + \text{Analysis}.$$

E.g. we have games that characterise the Baire property, Lebesgue measurability, etc.

Gale and Stewart (1953) reinvented inf games.

In 1960, Mycielski visited California to meet Blackwell (of whom you have heard in IB Stats) et al at Berkeley, and proposed the Axiom of Determinacy AD (then a.k.a. Axiom of *Determinateness*) as an alternative to (i.e. contradicts) the Axiom of Choice AC.

Then set theory is involved. Solovay worked on this, and in the late 1960s to 1980s, AD become a big area of set theory.

Definition 0.9 (Infinite games)**.** Let M be the set of **moves**. We play by moving alternatingly, i.e. I plays an element of M, then II, then I, and so on.

$$\begin{array}{lccccc} \text{I} & m_0 & & m_2 & & \\ & & & & & \cdots \\ \text{II} & & m_1 & & m_3 & \end{array}$$

We call finite sequence in M **positions** (i.e. $\in M^{<\omega}$), and infinite sequences **plays** (i.e. $\in M^\omega$).

$A \subseteq M^\omega$ is the **payoff set**. The **game** $G(A)$ is one s.t. I wins if $m = (m_0, m_1, \cdots) \in A$ and otherwise II wins.

Question 0.10. What if we have different rules for different players?

We shall see that the variant $G(A;T)$ of such games is equiv to one in the general form $G(A)$, i.e. $\forall A, T$, we can find A_T s.t. $G(A;T)$ is the same game as $G(A_T)$.

Definition 0.11 (Strategy)**.** A **strategy** is $\sigma : M^{<\omega} \to M$. *"from position to moves"*

If σ, τ are strats, then we can define by recursion a play $\sigma * \tau$ (" σ against τ") as follows:

If $\sigma * \tau \restriction n$ is defined (i.e. we already know the $0, 1, \cdots, n-1$ -th moves), then define

$$\sigma * \tau(n) = \begin{cases} \sigma(\sigma * \tau \restriction n) & \text{if } n \text{ is even,} \\ \tau(\sigma * \tau \restriction n) & \text{if } n \text{ is odd.} \end{cases}$$

σ is **winning for I** in $G(A)$ if $\forall \tau,\ \sigma * \tau \in A$. Simly τ is **winning for II** in $G(A)$ if $\forall \sigma,\ \sigma * \tau \notin A$.

Remark 0.12. Only half of σ is useful for a player but there is no harm in having answers for questions that you will never ask.

$\mathcal{P}(M^\omega)$ is the set of all games.

Question 0.13. Is there any M s.t. $\mathcal{P}(M^\omega)$ has a game that has no win strat for I and no win strat for II?

§1 Bonus: Combinatorial Games

This section is by guest lecturer Prof Imre Leader and is non-examinable.

Definition 1.1. Let X be a set (infinite or otherwise), and let $H \subseteq \mathcal{P}(X)$ be a collection of *finite* subsets of X (e.g. winning lines in noughts-and-crosses.)

The **combinatorial game/hypergraph game** on H has two players I and II, who take turns to claim unclaimed points in X, with I first. The first player to make a set in H wins. If the game ends and no one wins, it is a **draw**. The game **ends** if the board is full for X finite, or if players generate a sequence of length ω with no win for X infinite.

Remark 1.2. This is not kombo games in the Conway sense (nim, hackenbush etc. See *Winning Ways for your Mathematical Plays*)

Example 1.3

Some kombo games are:

1. noughts-and-crosses, aka tic-tac-toe in AmE. Or on $3 \times 3 \times 3$ or $4 \times 4 \times 4$ cubes.
2. Binary tree game B_n. Let X be the set of nodes on a perfect binary tree of depth n, H be the set of all paths from root to a leaf. I wins.
3. Ramsey game (K_N, K_S). $X = E(K_N)$, H is the set of all copies of K_S in K_N.
4. n-in-a-row. $X = \mathbb{Z}^2$, H is the set of n consecutive points, horizontal, vertical, or diagonal.

In any hypergraph game, exactly one of the following holds:

- I has win strat,
- II has win strat,
- both have drawing strat.

This is trivial if X finite by backtracking from the leaves of the game tree.

Proposition 1.4

Suppose I has no win strat, then II has a draw strat (s.t. II either draws or wins).

Proof. II draws or wins as follows: I plays some move x_0, then the current position is not forced win for I, so II has some move x_2 after which I does not have a forced win. Play this move. Repeat.

We obtain a play of the game in which at each time I does not have a win strat, so in particular there is no time when I wins by completing a winning line. *Winning lines are finite,* so at the end of this play, I does not complete a winning line. □

Remark 1.5. Our games are **open games** where if you win, then you have some time when you know you win.

Proposition 1.6 (Strategy stealing)
No kombo game is II win.

Proof. Suppose II has a win strat τ. We shall win as I, which is a contradiction.
Play some x_0 and pretend we have not played there and follows the strat τ of II. If τ calls us to take x_0, then we instead take x_1 and forget, and remember x_0. Repeat.
Then in the game in mind, we win first, and so on the board we win as the extra move cannot hurt us. □

Remark 1.7. We have:

1. Proof is completely non-constructive.
2. Equivly, if the game has no draw, then it is a I win.

Example 1.8
Recall Example 1.3. Which are I wins?

1. Noughts-and-crosses is draw, $3 \times 3 \times 3$ is I win (easy), $4 \times 4 \times 4$ is I win (by computer only, no human strat known).
2. Binary tree games. I wins.
3. Ramsey games. No draw if $N \geq R(s)$, the **Ramsey number**, so I wins. We know $\sqrt{2}^s \leq R(s) \leq 4^s$. In the case $S = 3$, the **$\triangle$-game**, I wins easily: claim edge 12, then II claims 13 or 34 up to relabeling. I claims 15. II is forced to respond by 25. I claims 16 and has two potential K_3 in one move: 126 and 156.

   In the case $S = 4$, no win strat is known. This is hard because we must make K_S and block opponent. Just like chess, we do not win by making checks as much as possible, but by building up subtle attacks.
4. n-in-a-row. $n \geq 8$ is draw. $n \leq 4$ is I win.

Warning 1.9. A game can be I win but not in bounded time, even if winning lines have bounded size and degrees are bounded, i.e. each point is in $\leq K$ winning lines for some const K.

Example 1.10
Consider the hypergraph on $[2n+1]$, with edge set $\{e_k = \{2k-1, 2k, 2k+1\} : k \in [n]\}$. Call this game T_n. If II makes two consecutive moves by claiming 0 and 2, then I is forced to claim 1, and then II can claim 4, and I is forced to claim 3 and so on.
Let the game be $G = B_4 + T_1 + T_2 + \cdots$, where the sum means in each turn a player chooses exactly one summand and make a move therein, and wins by winning one of the summand. Can check I wins but not in bounded time.

Problem 1.11 (Open)**.**

Q1. Could 5-in-a-row be I win but not in bounded time?

Q2. Known (K_N, K_S) is I win for large N. Is (K_ω, K_S) I win? Note that it is possible for a game to be a draw even if it is I win on all finite subboards, e.g. $B_4 + \{*,*\} + \{*,*\} + \cdots$, where $\{*,*\}$ is the game in which one wins by taking both points.

Q3. Is (K_N, K_S) win in bounded time for N vary and S fixed?

Exercise 1.12. Show that Q2 above is equivalent to Q3.

§2 Basics

§2.1 Strategies and Trees

Definition 2.1. $A \subseteq M^\omega$ is **determined** if either I or II has a win strat in $G(A)$.

Definition 2.2. A set $T \subseteq M^{<\omega}$ is a **tree** if it is closed under initial segments, i.e. if $p \in T$ and $q \subseteq p, q \in M^{<\omega}$, then $q \in T$.

To justify our $q \subseteq p$ notation, $p \in M^{<\omega}$ means $p : U \to M$ for some $m \in \omega$, so p is a set $p = \{(0, m_0), (1, m_1), \cdots, (u-1, m_{u-1})\}$.

Definition 2.3. If $s \in M^{<\omega}$ and $x \in M^\omega$, then we write sx for the **concatenation**

$$sx(n) = \begin{cases} s(n), & \text{if } u < |s| \\ x(n - |s|), & \text{if } u \geq |s|. \end{cases}$$

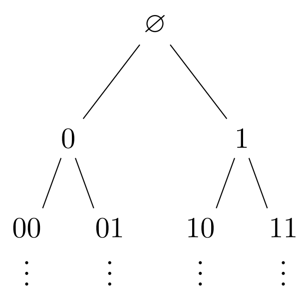
If s is a seq of length 1 with element m, then we write mx for sx.

We use the usual (by def, set-theoretic) notation for restriction

$$x \restriction n = \{(k, x(k)) \in x : k < n\},$$

so if $x \in M^\omega$, then $x \restriction n \in M^{<\omega}$. If $s \in M^{<\omega}$, $x \in M^\omega$, then $sx \restriction |s| = s$.

A tree here has a *uniquely identified* root $\varnothing$ unlike graph-theoretic trees. For $M = 2 = \{0, 1\}$, the whole tree $M^{<\omega}$ looks like:



Definition 2.4. If T is a tree, then we say $x \in M^\omega$ is a **branch** through T if $\forall n \in \omega,\ x \restriction n \in T$.

We write $[T] = \{x \in M^\omega : x \text{ is a branch through } T\}$ for the **set of branches through** T.

Remark 2.5. If x is not a branch of T, then we know after finite time that it is not.

Definition 2.6. If σ is a strat, then we shall def the **I-strategic tree** T^{I}_σ of σ and the **II-strategic tree** T^{II}_σ of σ.

Assume we play as I with σ, then we start with $\sigma(\varnothing)$, and II has many moves, and we have a unique response, so we def $p \in T_\sigma^{\mathrm{I}}$ if $\forall n$ s.t. $2n \in \operatorname{dom} p$, we have $p(2n) = \sigma(p \restriction 2n)$.

Simly, $p \in T_\sigma^{\mathrm{II}}$ iff $\forall n$ s.t. $2n+1 \in \operatorname{dom} p$, we have $p(2n+1) = \sigma(p \restriction 2n+1)$.

$\sigma * \tau$ has the property that $\forall n$,

$$\sigma * \tau \restriction n \in T_\sigma^{\mathrm{I}}, \quad \sigma * \tau \restriction n \in T_\tau^{\mathrm{II}}$$

by def and induction. So $\sigma * \tau \in [T_\sigma^{\mathrm{I}}] \cap [T_\tau^{\mathrm{II}}]$. In fact it is the only element (easy question on Sheet 1).

Definition 2.7. Fix $x : \omega \to M$ and define the **blindfolded strategies** σ_x and τ_x by:

$$\sigma_x(p) = x(n) \text{ if } |p| = 2n,$$
$$\tau_x(p) = x(n) \text{ if } |p| = 2n+1.$$

I.e. the players only know which turn but does not sense other things about the position.

Definition 2.8. (I-part and II-part) If $x \in M^\omega$, then $x_{\mathrm{I}}(n) = x(2n)$, and $x_{\mathrm{II}}(n) = x(2n+1)$. So $(\sigma * \tau_x)_{\mathrm{II}} = x, (\sigma_x * \tau)_{\mathrm{I}} = x$.

Proposition 2.9

$$[T_\sigma^{\mathrm{I}}] = \{\sigma * \tau : \tau \text{ strat}\}.$$

Proof. We have seen $\supseteq$. Conversely, if $x \in [T_\sigma^{\mathrm{I}}]$, then let $z = x_{\mathrm{II}}$, and consider τ_z blindfolded. Then $\sigma * \tau_z = x$ by induction. $\square$

Proposition 2.10

σ is w.s. for I iff $[T_\sigma^{\mathrm{I}}] \subseteq A$, and simly τ is w.s. for II iff $[T_\tau^{\mathrm{II}}] \subseteq M^\omega \setminus A$.

So we may have ugly sets that is a I win. I-win is a local property that as long as A includes the branch set of a strat tree, then $G(A)$ is I-win.

Theorem 2.11

If $|M| \geq 2$, then

1. If A is a win for I, then $|A| \geq 2^{\aleph_0}$.
2. If A is a win for II, then $|M^\omega \setminus A| \geq 2^{\aleph_0}$.

Proof. By the previous observation, WTP $\left|[T_\sigma^{\mathrm{I}}]\right| \geq 2^{\aleph_0}$. We shall construct an injection from 2^ω to $[T_\sigma^{\mathrm{I}}]$. Let $m_0 \neq m_1 \in M$. If $x \in 2^\omega$, then define

$$\begin{aligned} \bar{x} : \quad \omega &\to M \\ \bar{x}(n) &= m_{x(n)}. \end{aligned}$$

Now define

$$\begin{aligned} f : \quad 2^\omega &\to [T_\sigma^{\mathrm{I}}] \\ f(x) &= \sigma * \tau_x. \end{aligned}$$

This is an injection since II makes different moves in $\sigma * \tau_x$ and $\sigma * \tau_y$. $\square$

We have seen a necessary condition for I-win and II-win, but this condition is trivial if we look at the disjunction (i.e. this tells us nothing about determined sets), because $\forall A$, either $|A|$ or $|M^\omega \setminus A|$ is $\geq 2^{\aleph_0}$.

Theorem 2.12

If $|M| \geq 2$ and A ctble, then II wins. If $|M| \geq 2$ and $M^\omega \setminus A$ ctble, then I wins.

Proof. If A ctble, then write $A = \{a_u : u < \omega\}$. Use a diagonal argument, i.e. as II, we pick

$$d(i) = \begin{cases} m_0, & \text{if } a_i(2i+1) \neq m_0, \\ m_1, & \text{o/w.} \end{cases}$$

Then the blindfolded τ_d wins against any σ for I. □

§2.2 Zermelo's Theorem and the Gale–Stewart Theorem

This leaves the question of sets A with both A and $M^\omega \setminus A$ unctble. $|A|$ is not a promistion criterion. We need to look at something else.

Definition 2.13. $G(A)$ is **finite** (as an infitite game) if $\exists n \leq \omega$ s.t. whether $x \in A$ depends only on $x \upharpoonright n$, i.e. $\forall x, y$ with $x \upharpoonright n = y \upharpoonright n$, $x \in A \iff y \in A$. "*You can continue to play but that is a waste of an infinite amount of time.*"

Finite games allows for **backwards induction**. Human rationality seems to work on this by foreseeing possible outcomes but this is rubbish because computation is exponentially expensive.

Theorem 2.14 (Zermelo's, retrospectively in his paper for chess)

Finite games are determined.

Proof. Label the game tree by recursion. Labels are I and II. At time n (this is a horizontal line in the game tree, and is orange on blackboard), we know who wins. If $|p| = n$, then label p by $\begin{cases} \text{I}, & \text{if } \forall x,\ p \subseteq x \implies x \in A, \\ \text{II}, & \text{if } \forall x,\ p \subseteq x \implies x \notin A. \end{cases}$ By induction, assume all pos of length $k+1$ have been labelled, and now we label those of length k. If k even, then I plays at that pos p.

Case 1. If $\exists$ successor q of length $k+1$ s.t. q is labelled I, then label p by I.

Case 2. If $\forall$ successor q of length $k+1$, q is labelled II, then label p by II.

Simly for k odd. This produces a total labelling $\ell : M^{<\omega} \to \{I, II\}$ by extending below the orange line in the obvious way.

Claim 2.15. If $\ell(\varnothing) = \text{I}$, then I wins, and vice versa.

Proof. The win strat is "*Stay on you label*". □

□

Remark 2.16. A.k.a. Zermelo's trivial observation (by Leader).

Remark 2.17. We might need the Axiom of Ctble Choice **CC** to get from a mental recipe to a strat (which is a function) even in the simplest case where $A = M^\omega$. The proof specifies a subtree of label I (called **quasi-strategy**). If we want to extract a genuine strategy, then we need some way of picking $\sigma(p)$. This is in particular the case if AC holds or M is well-ordered (e.g. $M = \omega$).

Chess is not of the form $G(A)$ because there are draws. Instead we take $A = \{x : \exists n \text{ s.t. } x \restriction n \text{ is winning for white}\}$, i.e. any draw is considered a black win.

Definition 2.18. A is **finitary** if there is a set $S \subseteq M^{<\omega}$ of positions s.t. $x \in A \iff \exists n,\ x \restriction n \in S$.

This means wins for I are determined after finitely many steps, but with no bounds on the lengths, and not necessarily so for II.

Theorem 2.19 (Gale–Stewart, but known by the Polish e.g. Banach, Ulam et al)
Finitary games are determined.

Proof. Now the orange line is broken into pieces and some branch does not cross the line at all. IDEA: throw away II for now. We do a partial labelling

$$\ell : M^{<\omega} \dashrightarrow \{\mathrm{I}\}$$

and label everything else II and hope it works.

We do this by recursion as before. Write $\ell_0(p) = \mathrm{I}$ iff $\exists n$ s.t. $p \restriction n \in S$.

<u>Recursion step</u>

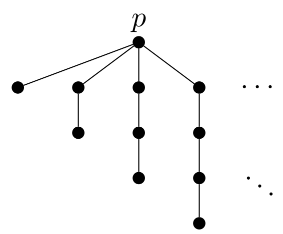
Case 1a. If $|p|$ even, and all successors of p are not labelled, then not label.

Case 1b. If $|p|$ even, and there is a successor labelled I, then label I.

Case 2a. If $|p|$ odd, and all successors labelled I, then label I.

Case 2b. If $|p|$ odd, and there is a successor not labelled, then not label.

But we cannot assumn that all label statuses of the successors are already fixed. Def $\ell_{\alpha+1}$ to be the extension of ℓ_α that labels previously unlabelled pos according to our recursion step. If M is infinite, there might be pos p s.t. $\forall q$ successor of p, $\exists n$ s.t. $\ell_n(q) = \mathrm{I}$, but there is no N s.t. in step N all succ are labelled.



So we need transfinite induction. For λ non-zero limit, let

$$\ell_\lambda = \bigcup_{\alpha<\lambda} \ell_\alpha$$

WTP recursion ends and hope comes true. Say that an ordinal α is a **fixed point** of the G–S labelling if $\ell_{\alpha+1} = \ell_\alpha$.

Claim 2.20. There is a fixed point, i.e. recursion ends.

Proof. Let $D_\alpha = \mathrm{dom}(\ell_{\alpha+1}) \setminus \mathrm{dom}(\ell_\alpha)$. If $\alpha \neq \beta$, then $D_\alpha \cap D_\beta = \varnothing$. Also, $D_\alpha = \varnothing$ iff α is a fixed point. So if α is not a fixed point, then

$$\begin{aligned} D: \quad & \alpha \to \mathcal{P}(M^{<\omega}) \\ & \beta \mapsto D_\beta \end{aligned}$$

is an injection (images are non-empty and disjoint), but by Hartogs' lemma, $\forall X$, $\exists \alpha$ ordinal (called **Hartogs' $\aleph$** of X) s.t. α does not inject to X, so α for $\mathcal{P}(M^{<\omega})$ is a fixed point. □

Def $\ell^*(p) = \mathrm{I}$ iff $\exists \beta < \alpha$ s.t. $\ell_\beta(p) = \mathrm{I}$, and def

$$\ell(p) = \begin{cases} \mathrm{I}, & \text{if } p \in \mathrm{dom}(\ell^*), \\ \mathrm{II}, & \text{o/w.} \end{cases}$$

If $p \in \mathrm{dom}(\ell^*)$, let $\mathrm{age}(p)$ be the least ord γ s.t. $p \in \mathrm{dom}(\ell_\gamma)$.

Claim 2.21. If $\ell(\varnothing) = \mathrm{I}$, then I has a w.s. and vice versa.

Proof. The win strat for II is "*Stay on label II*". Suppose $x \in M^\omega$ follows this strat, then $\forall n,\ \ell(x \restriction n) = \mathrm{II}$, so $\ell(x \restriction n) \neq \mathrm{I}$, so $\ell(x \restriction n) \notin S$. This says $x \notin A$.

The win strat for I is "*Stay on label I and strictly decrease age*". Suppose $x \in M^\omega$ follows this strat, then $\alpha_n = \mathrm{age}(x \restriction n), n \leq \omega$ is a seq of strictly decreasing ords, so it must hit 0, i.e. $\exists m$ s.t. $\mathrm{age}(x \restriction m) = 0$, so $\ell_0(x \restriction m) = \mathrm{I}$, so $\exists n \leq m,\ x \restriction n \in S$ by def of ℓ_0. Hence $x \in A$. □

This gives Theorem 2.19. □

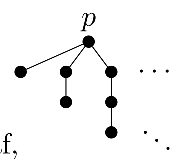
We make a small detour to the necessity of transfinite recursion in the proof. Recall $[T]$ is the set of branches through T.

Definition 2.22. T is **well-founded (wellfdd)** if $[T] = \varnothing$.

Lemma 2.23 (König's)
If M is finite, then any wellfdd tree on M is finite.

Proof. If T is infinite, then one subtree is infinite. Continue going down to obtain a branch. □

If M is infinite, then König's lem does not hold. Recall the example below which we have seen. This is wellfdd but not finite.



Definition 2.24. The **height** of a node is

$$\mathrm{ht}(p) = \begin{cases} 0, & \text{if } p \text{ is a leaf}, \\ \sup\{\mathrm{ht}(q) + 1 : q \text{ is an immediate successor of } p\}, & \text{o/w.} \end{cases}$$

Say $\mathrm{ht}(T) = \mathrm{ht}(\varnothing)$, i.e. the height of the root.

$\forall \alpha < |M|^+$, there is a wellfdd tree on M with $\mathrm{ht}(T) = \alpha$. E.g. given a tree of height ω, just add one new vertex above the root to obtain $\omega + 1$. This is the reason that we need transifinite induction in the proof.

Theorem 2.25

Assume AC, then there are non-determined sets if $|M| \geq 2$.

Proof. For notational convenience we shall only do $M = \omega$, other cases being same. We will diagonal-argue against all strategic trees.

If T is a strat tree, then $T \subseteq \omega^{<\omega}$, so T is ctble, so $|[T]| \leq 2^{\aleph_0}$. But we have seen $\geq$ in the proof of Theorem 2.11. Hence $|[T]| = 2^{\aleph_0}$.

How many strat trees? If $x \neq y \in \omega^\omega$, then $T^{\mathrm{I}}_{\sigma_x} \neq T^{\mathrm{I}}_{\sigma_y}$, so there are at least $2^{\aleph_0}$ many. But $T \subseteq \omega^{<\omega}$, so there are exactly $2^{\aleph_0}$.

We shall def by recursion $A_\alpha, B_\alpha \subseteq \omega^\omega$ s.t. $|A_\alpha| = |B_\alpha| = |\alpha|$. Start with $A_0 = B_0 = \varnothing$, then if A_α and B_α are defined, then consider the α-th strat tree (we list them as T_α, $\alpha < 2^{\aleph_0}$).

Since $|[T_\alpha]| = 2^{\aleph_0}$,

$$|[T_\alpha] \setminus (A_\alpha \cup B_\alpha)| = 2^{\aleph_0}.$$

By AC, pick $a_\alpha \neq b_\alpha \in [T_\alpha] \setminus (A_\alpha \cup B_\alpha)$, and let

$$\begin{aligned} A_{\alpha+1} &= A_\alpha \cup \{a_\alpha\}, \\ B_{\alpha+1} &= B_\alpha \cup \{b_\alpha\}. \end{aligned}$$

This satisfies $|A_{\alpha+1}| = |B_{\alpha+1}| = |\alpha + 1|$, and $\bigcup_\alpha A_\alpha$ is disjoint from $\bigcup_\alpha B_\alpha$. Def $A = \bigcup_{\alpha < 2^{\aleph_0}} A_\alpha$.

Claim 2.26. A is not determined.

Proof. If it is, then either there is a I-strat tree T_α s.t. $[T_\alpha] \subseteq A$, or a II-strat tree T_β s.t. $[T_\beta] \subseteq \omega^\omega \setminus A$. WLOG $[T_\alpha] \subseteq A$, then $b_\alpha \in [T_\alpha]$ and $b_\alpha \in B_{\alpha+1}$. But $B_{\alpha+1}$ is disjoint from A, so b_α is not in A, contradiction. □

□

§3 The Baire Space and Pointclasses

We hoped that "Every simple set is determined". Our first attempt is a criterion based on size, but not very helpful. And by AC, there are non-determined sets.

Second attempt: Def simplicity by "topological complexity". For now fix $M = \omega$. (This is not an innocent choice: something really depends on this, e.g. well-orderedness, but it is a natural choice.)

ω^ω is a product $\prod_{n \in \omega} \omega$, so it has a natural (product) topology inherited from ω. Besides, for $p \in \omega^{<\omega}$, we def $[p] = \{x \in \omega^\omega : p \subseteq x\}$. These cylinder sets form a base of a topology. So $A \subseteq \omega^\omega$ is open iff $\exists S \subseteq \omega^{<\omega}$ s.t. $A = \bigcup_{p \in S} [p]$. In other words, A is open iff A is finitary.

Proposition 3.1 (properties of ω^ω)

1. If we pick the discrete topology on ω, then the topology above is the prod topology on ω^ω, i.e. the weakest s.t. the projections $\pi_i : \omega^\omega \to \omega,\ x \mapsto x_i$ are cts.

2. If $x \neq y \in \omega^\omega$, then we define $d(x, y) = 2^{-n}$ where n is the min number s.t. $x \restriction n \neq y \restriction n$, and $d(x, x) = 0$. This is a metric on ω^ω, and the balls are

$$B_{\frac{1}{2^n}} = \{y : y \restriction n = x \restriction n\} = [x \restriction n],$$

   which are precisely the basic open sets above, so ω^ω is a metric space.

3. Any $[p]$ is also closed, i.e. they are clopen, because

$$\omega^\omega \setminus [p] = \bigcup_{\substack{q \neq p \\ |q| = |p|}} [q].$$

   Such spaces (with a base of clopen sets) are called **zero-dimensional**, and are **totally disconnected**, i.e. the only connected subsets are $\varnothing$ (? depending on your def) and singletons.

4. ω^ω is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

   Proof. $x \in \omega^\omega \mapsto [x_0 + 1, x_1 + 1, \cdots]$ (continued fractions) is a homeomorphism from ω^ω to $(1, +\infty) \setminus \mathbb{Q}$. □

   "Uses maths not [*sic*] taught these days but were taught in the 19th century."

5. Convergence can be described via metric:

$$(x_n) \to x \iff \forall k,\ \exists N \text{ s.t. } \forall n > N,\ d(x_n, x) < 2^{-k}, \text{ i.e. } x_n \restriction k = x \restriction k.$$

   So x_n agrees with x along longer and longer prefixes.

Let's look at closed sets. Suppose T is a tree on ω.

Lemma 3.2
$[T]$ is closed.

Proof. Suppose $x_n \in [T]$, and $(x_n) \to x$. Then $\forall k,\ x_n \restriction k \in T$, and $\forall k,\ \exists n$ s.t. $x_n \restriction k = x \restriction k$, so $x \restriction k \in T$, so $x \in [T]$. □

If $A \subseteq \omega^\omega$, then def $T_A = \{x \restriction n : x \in A \text{ and } n \in \omega\}$. So T_A is a tree, i.e. closed under initial segments.
By the lem, $[T_A]$ is closed. If $x \in A$, then $x \in [T_A]$, so $A \subseteq [T_A]$.

Lemma 3.3
If A is closed, then $A = [T_A]$.

Proof. Suppose $z \in [T_A]$, then $\forall n,\ z\restriction n \in T_A$. So $\forall n,\ \exists x_n \in A$ s.t. $z\restriction n = x_n\restriction n$. Now $(x_n) \to z$, so $z \in A$. □

This says $[T_A] = \mathrm{Cl}(A)$ is the closure.

Corollary 3.4 (Tree representation thm of closed sets)
A is closed iff $\exists T$ tree s.t. $A = [T]$.

Easy but is "*mother of all tree rep thms*".

§3.1 The Borel Hierarchy

Question 3.5. How many open sets are there?

We have ctbly many basic open sets, so at most $2^{\aleph_0}$ open sets (and in fact precisely that many).

Question 3.6. How many closed sets?

Same as the open sets (by taking complements). Alternatively, as many as trees, so $2^{\aleph_0}$.

Gale–Stewart shows that every open set is determined, but these are few: we have $|\mathcal{P}(\omega^\omega)| = 2^{2^{\aleph_0}}$ choices for the payoff set A.

On Sheet 1, we shall show that difference $A \setminus B$ of open sets is determined, by the modified proof of Gale–Stewart. But there are other operations.

Definition 3.7. A **$\boldsymbol{\sigma}$-algebra** is a set family closed under complementation and ctble union.

The family of open sets are closed under ctble union, but not necessarily under complementation (unless open = closed). Closed sets are simly closed under ctble intersection.

Definition 3.8. Closure of open sets under ctble intersections is $\boldsymbol{G_\delta}$, and the σ-closure thereof is $\boldsymbol{G_{\delta\sigma}}$ etc. Simly, closure of closed sets under ctble union is $\boldsymbol{F_\sigma}$, and the δ-closure thereof is $\boldsymbol{F_{\sigma\delta}}$ etc.

Definition 3.9 (Borel hierarchy)**.** The **Borel $\boldsymbol{\sigma}$-algebra** is the smallest σ-alg containing the open sets. We can analyse this in the form of a hierarchy (**Borel hierarchy**).

Fix a topological space X. Def:

$$\boldsymbol{\Sigma}^0_1 = \{A \subseteq X : A \text{ open}\}.$$

If $\boldsymbol{\Sigma}^0_\xi$ is defined, then def

$$\boldsymbol{\Pi}^0_\xi = \{X \setminus A : A \in \boldsymbol{\Sigma}^0_\xi\}.$$

If $\forall \alpha < \xi,\ \boldsymbol{\Pi}^0_\alpha$ is defined, then

$$\boldsymbol{\Sigma}^0_\xi = \left\{ \bigcup_{n\in\omega} A_n : \forall n,\ A_n \in \bigcup_{\alpha<\xi} \boldsymbol{\Pi}^0_\alpha \right\}.$$

Also write

$$\boldsymbol{\Delta}^0_\xi = \boldsymbol{\Sigma}^0_\xi \cap \boldsymbol{\Pi}^0_\xi.$$

This is good notation when we go beyond ω iterations, because we do not need to write $G_{(\delta\sigma\delta\cdots)\delta\sigma}$ etc.

$\forall\alpha$, $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathcal{P}(X)$, so by Replacement, this has a fixed point (by Hartogs' lem), i.e. $\mathbf{\Sigma}^0_\alpha = \mathbf{\Pi}^0_\alpha$ for some α.

Question 3.10. Is this a 'hierarchy' in the sense that $\mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Sigma}^0_\eta$ for $\xi \leq \eta$?

Question 3.11 (warm up)**.** Is every open set a ctble union of closed sets?

In ω^ω, this follows from 0-dimensionality: every open set is a ctble union of basic clopen sets. In $\mathbb{R}$, it still works but only with a different proof. In general, this is FALSE!

But anyway $\mathbf{\Sigma}^0_1(\omega^\omega) \subseteq \mathbf{\Sigma}^0_2(\omega^\omega)$. We also get $\mathbf{\Pi}^0_\xi \subseteq \mathbf{\Sigma}^0_{\xi+1}$ for free, so $\mathbf{\Pi}^0_\xi \subseteq \mathbf{\Sigma}^0_\eta \quad \forall\xi < \eta$.

By taking complements, this also says $\mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Pi}^0_{\xi+1}$, so $\mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Pi}^0_\eta \quad \forall\xi < \eta$. It remains to show $\mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Sigma}^0_\eta$ for $\xi < \eta$. We use induction. Suppose $\forall\beta < \alpha < \eta$, $\mathbf{\Sigma}^0_\beta \subseteq \mathbf{\Sigma}^0_\alpha$, then also $\mathbf{\Pi}^0_\beta \subseteq \mathbf{\Pi}^0_\alpha$.

Now take $A \in \mathbf{\Sigma}^0_\xi$. Then

$$A = \bigcup_{n\in\omega} A_n, \quad A_n \in \mathbf{\Pi}^0_{\alpha_n}$$

for some $\alpha_n < \xi$. Since $\xi < \eta$, we have $\forall n,\ \alpha_n < \eta$, so $A \in \mathbf{\Sigma}^0_\eta$.

Remark 3.12. We are only using the def but not the induction hypothesis, i.e. this proof is badly written.

Therefore $\mathbf{\Pi}^0_\xi \cup \mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Sigma}^0_{\xi+1}$, so taking complements $\mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi \subseteq \mathbf{\Pi}^0_{\xi+1}$, so

$$\mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi \subseteq \mathbf{\Delta}^0_{\xi+1}.$$

In diagram this looks like

$$\begin{tikzcd}
 & \mathbf{\Sigma}^0_1 \arrow[dr, hook] \arrow[dd, dash, dotted, "\neq" description] & & \mathbf{\Sigma}^0_2 \arrow[dr, hook] & & \cdots & & \mathbf{\Sigma}^0_\omega \arrow[dr, hook] & \\
\mathbf{\Delta}^0_1 \arrow[ur, hook] \arrow[dr, hook] & & \mathbf{\Delta}^0_2 \arrow[ur, hook] \arrow[dr, hook] & & \mathbf{\Delta}^0_3 \arrow[ur, hook] \arrow[dr, hook] & & \mathbf{\Delta}^0_\omega \arrow[ur, hook] \arrow[dr, hook] & & \mathbf{\Delta}^0_{\omega+1} \cdots \\
 & \mathbf{\Pi}^0_1 \arrow[ur, hook] & & \mathbf{\Pi}^0_2 \arrow[ur, hook] & & \cdots & & \mathbf{\Pi}^0_\omega \arrow[ur, hook] &
\end{tikzcd}$$

We have seen that this stops at some α, and we say that "*the hierarchy collapses*".

Proposition 3.13

$\aleph_1$ is an upper bound of the stopping ordinal, i.e.

$$\mathbf{\Sigma}^0_{\aleph_1} = \bigcup_{\alpha<\aleph_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\aleph_1} \mathbf{\Pi}^0_\alpha.$$

Proof. We already know $\supseteq$. WTP $\subseteq$. Take $A \in \mathbf{\Sigma}^0_{\aleph_1}$. By def, $A = \cup_{n<\omega} A_n$, where $A_n \in \mathbf{\Pi}^0_{\alpha_n}$ and $\alpha_n < \aleph_1$.

By regularity of $\aleph_1$, $\alpha = \bigcup_{n\in\omega} \alpha_n < \aleph_1$. □

Remark 3.14. Regularity of $\aleph_1$ is not a thm of ZF, but of $\mathsf{ZF} + \mathsf{AC}_\omega(\omega^\omega)$, i.e. we need choice from ω many subsets of ω^ω.

Sometimes the hierarchy collapses much earlier. Let X be a ctble metric space, then $\forall x \in X$, $\{x\} \in \mathbf{\Pi}^0_1(X)$. So $A = \bigcup_{x\in A}\{x\}$ is a ctble union of closed sets, so $A \in \mathbf{\Sigma}^0_2$. This says

Proposition 3.15

If X ctble, then $\mathcal{P}(X) = \mathrm{Borel}(X) = \mathbf{\Delta}^0_2(X)$.

Question 3.16. In ω^ω, what is a $\mathbf{\Sigma}^0_2$ set that is neither open nor closed?

Paradigmatic: Take

$$A = \{x \in \omega^\omega : \exists n\ \forall k \geq n,\ x(k) = 0\},$$

the set of eventually zero seqs.

$\underline{A \in \mathbf{\Sigma}^0_2.}$

$$A = \bigcup_{n\in\omega} A_n \text{ for } A_n = \{x : \forall k \geq n,\ x(k) = 0\}.$$

Here A_n is the branch set of a tree, whence closed.

<u>A not open.</u> It cannot contain any basic open sets.

<u>A not closed.</u> Take $x_n = \underbrace{111\cdots1}_{n}0\cdots \in A_n$, then $(x_n) \to \mathbf{1}$, but $\mathbf{1} \notin A$.

Remark 3.17. We spend a long time proving $\mathbf{\Delta}^0_2 \supsetneq \mathbf{\Delta}^0_1$. But Lent term is too short to repeat this up to $\aleph_1$. We need another technique to prove a non-collapsing thm, namely the technique of **universal sets**.

Every $\mathbf{\Sigma}^0_1$ is determined by Gale–Stewart.

$\mathbf{\Sigma}^0_2$? This cannot be done by constructive labelling: after n steps in $G(A)$ where $A = \{x \in \omega^\omega : \exists n\ \forall k \geq n,\ x(k) = 0\}$, the game still looks the same. Yet II wins $G(A)$ by playing 1 infinitely often.

Fact 3.18 (History of Borel determinacy)**.**

a. 1953, $\mathbf{\Sigma}^0_1$, Gale–Stewart.

b. 1955, $\mathbf{\Sigma}^0_2$, Wolfe.

c. 1963, $\mathbf{\Sigma}^0_3$, Davis (controversy over Polish work 1960–1961, but usually people cite Davis).

d. 1972, $\mathbf{\Sigma}^0_4$, Paris used set theoretic instead of kombo arguments.

e. 1970–1971, H. Friedman showed for n large, determinacy of $\mathbf{\Sigma}^0_n$-sets cannot be proved without "substantial" set theory.

f. 1975, Borel alg, Martin (usually cited when people apply inf games, but overkill because $\mathbf{\Sigma}^0_4$ are already very complex to actually show up in application.)

§3.2 Universal Sets

Definition 3.19 (Universal sets)**.** Let X, Y be topological spaces, $\mathbf{\Gamma}$ be $\mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$ or $\mathbf{\Delta}^0_\alpha$. Think of $\mathbf{\Gamma}(Y)$ as being parametrised by elements of X. We call $U \subseteq X \times Y$ **universal** if

(i) U is $\mathbf{\Gamma}(X \times Y)$,

(ii) $\forall A \in \mathbf{\Gamma}(Y),\ \exists x \in X$, s.t.

$$A = U_x = \{y \in Y : (x, y) \in U\}.$$

This is called a **section**.

Definition 3.20. An assignment $\mathbf{\Gamma}$ of subsets of a topological space X to X is called a **pointclass**, e.g. $X \mapsto \{U \subseteq X : U \text{ open in } X\}$. We call a pointclass **boldface** if it is closed under cts preimage, i.e. if $f : X \to Y$ is cts and $A \in \mathbf{\Gamma}(Y)$, then the pullback $f^{-1}(A) \in \mathbf{\Gamma}(X)$.

Remark 3.21. Historically boldface are typeset in boldface and markings on correction sheets $\underline{A}$ means *italics* and $\underset{\sim}{A}$ means **boldface**.

Definition 3.22. $\mathbf{\Gamma}$ is **coherent** if $\forall X \in \mathbf{\Gamma}(Y)$ with the subspace tplgy on X, we have $\mathbf{\Gamma}(X) = \{A \cap X : A \in \mathbf{\Gamma}(Y)\}$.

Also there are several closure properties whose meaning should be clear, e.g. **closed under complement**, **closed under ctble union**.

Example 3.23 (pointclasses)
These are all boldface:

1. $\mathbf{\Sigma}^0_\xi$ is closed under finite intersections, ctble unions, but in general not complements.
2. $\mathbf{\Pi}^0_\xi$ is closed under fin unions, ctble intersections, but in general not complements.
3. $\mathbf{\Delta}^0_\xi$ is closed under fin unions or intersections, and complements, but in general not ctble unions or intersections.

$\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ are coherent.

We shall make a short excursion by looking in detail at "$\mathbf{\Sigma}^0_2$ *is closed under ctble unions*".

Proof. Let $A = \bigcup_{n\in\omega} A_n$, $A_n \in \mathbf{\Sigma}^0_2$. Find C_{nm} s.t. $A_n = \bigcup_{m\in\omega} C_{nm}$, $C_{nm} \in \mathbf{\Pi}^0_1$.
Write
$$A = \bigcup_{\substack{n\in\omega \\ m\in\omega}} C_{nm},$$
which is a ctble union, so $A \in \mathbf{\Sigma}^0_2$. $\square$

Note that we have used a choice function to pick a representation $A_n = \bigcup C_{nm}$, as the def of $\mathbf{\Sigma}^0_2$ only says the set of such representations is non-empty.

Without using AC, $\mathbf{\Sigma}^0_2 \subseteq \mathbf{\Pi}^0_3$, so A is a ctble union of $\mathbf{\Pi}^0_3$ sets, so A is definitely in $\mathbf{\Sigma}^0_4$.

There is a famous model (Feferman–Lévy) of ZF + '$\mathbb{R}$ *is a ctble union of ctble sets*'. In this model, $\mathbb{R} = \bigcup_{n\in\omega} C_n$, with $C_n \subseteq \mathbb{R}$ ctble, so $\forall A \subseteq \mathbb{R}$,
$$A = \bigcup_{n\in\omega} (A \cap C_n),$$
and $A \cap C_n \subseteq C_n$ ctble, whence $\mathbf{\Sigma}^0_2$. Although this does not prove $A \in \mathbf{\Sigma}^0_2$, it proves $A \in \mathbf{\Sigma}^0_4$, so A is Borel, so the F–L models has no Lebesgue non-measurable sets.

Recall Definition 3.19 of universal sets.

Lemma 3.24 (universal set lemma)

Let $\mathbf{\Gamma}$ be a coherent boldface pointclass, $X \subseteq Y$ s.t. $X \in \mathbf{\Gamma}(Y)$, U be X-universal for $\mathbf{\Gamma}$. Then $\mathbf{\Gamma}$ is not closed under complements.

Proof. If $X \subseteq Y$, then some elements of $X \times Y$ are elmts of $X \times X$, so we can diag-argue.

Consider $x \mapsto (x, x)$, $X \to X \times Y$. This is cts. So

$$\{x \in X : (x, x) \in U\} \in \mathbf{\Gamma}(X) \quad \forall U \in \mathbf{\Gamma}$$

because this is the preimage of some boldface pointclass set.

Assume we have closure under complement, then

$$D = \{x \in X : (x, x) \notin U\} \in \mathbf{\Gamma}(X)$$

because $D = D \cap X \in \mathbf{\Gamma}(Y)$ and $\mathbf{\Gamma}$ is coherent. Universality gives some d s.t.

$$D = U_d = \{y : (d, y) \in U\}.$$

Now $d \in U_d \iff d \in D \iff d \notin U_d$, contradiction. □

IDEA: If we can show for $\alpha < \aleph_1$, $\exists X$-universal set for $\mathbf{\Sigma}^0_\alpha$, then the Boerl hierarchy does not collapse.

Theorem 3.25 (universal set thm)

$\exists U^\alpha \in \omega^\omega \times \omega^\omega$ that is ω^ω-universal for $\mathbf{\Sigma}^0_\alpha$.

Proof. Constunct by recursion. Look at $\alpha = 1$. What is an open set? $A \subseteq \omega^\omega$ is open iff $\exists S \subseteq \omega^{<\omega}$ s.t. $x \in A \iff \exists s \in S, s \subseteq x$.

Fix a bijection $s : \omega \to \omega^{<\omega}$. Then if $B \subseteq \omega$, then B represents the open set $\bigcup_{k \in B}[s(k)]$, and this exhausts all open sets.

Let now $x \in \omega^\omega$. Def $B_x = \{k \in \omega : x(k) \neq 0\}$ and $A_x = \bigcup_{k \in B_x}[s(k)]$. Say $(x, y) \in U$ iff $y \in A_x$. Then we have a parametrisation of open sets, i.e. (ii) in the def of universality is satisfied.

U satisfies (i). Suppose $(x, y) \in U$. To see that U is open, we shall find a basic open set P s.t. $(x, y) \in P \subseteq U$. If $(x, y) \in U$, then $y \in A_x = \bigcup k \in B_x[s(k)]$, so we have some k s.t. $x(k) \neq 0$ and $y \in [s(k)]$, so let $m = |s(k)|$, $N = \max(k + 1, m)$. If $(x', y') \restriction N = (x, y) \restriction N$, then $(x', y') \in U$, so U is open.

Suppose U is ω^ω-universal for $\mathbf{\Sigma}^0_\alpha$, then $\omega^\omega \times \omega^\omega \setminus U$ is $\mathbf{\Pi}^0_\alpha$, and if $A \subseteq \omega^\omega$ is $\mathbf{\Pi}^0_\alpha$, then $\omega^\omega \setminus A$ is $\mathbf{\Sigma}^0_\alpha$, so $\exists x$ s.t. $U_x = \omega^\omega \setminus A$, so

$$(\omega^\omega \times \omega^\omega \setminus U)_x = \omega^\omega \setminus U_x = A,$$

so $\omega^\omega \times \omega^\omega \setminus U$ is ω^ω-universal for $\mathbf{\Pi}^0_\alpha$.

Now it remains the case of $\mathbf{\Sigma}^0_\alpha$ $(\alpha < \aleph_1)$ and we can assume $\forall \beta < \alpha$, $\exists$ ω^ω-universal set U^β for $\mathbf{\Pi}^0_\beta$.

A generic $\mathbf{\Sigma}^0_\alpha$ set is $\bigcup_{n \in \omega} A_n$ for $A_n \in \mathbf{\Pi}^0_{\beta_n}$, $\beta_n < \alpha$ This means we have $x_n \in \omega^\omega$ s.t. $A_n = \left(U^\beta\right)_{x_n}$. We need to encode these x_n into an element of ω^ω. Consider $x \in \omega^\omega$. We can split ω into infinitely many infinite sets (e.g. the columns of $\omega \times \omega$, where we fix a bijection $b : \omega \times \omega \to \omega$). For each n, def the n-th subsequence $(x)_n$ by

$$(x)_n(m) = x(b(n, m)).$$

Now this is a natural bijection $(\omega^\omega)^\omega$ to ω^ω. Moreover, $x \mapsto (x)_n$ is cts $\forall n$, since finite initial segments of $(x)_n$ depend only on finite initial segments of x.

Since α is ctble, find a surjective $\pi : \omega \twoheadrightarrow \alpha$, s.t. for each $\beta < \alpha$, there are infinitely many $n < \omega$ s.t. $\pi(n) = \beta$. Let

$$U = \left\{(x, y) : \exists n,\ ((x)_n, y) \in U^{\pi(n)}\right\}.$$

Claim 3.26. This is universal for $\mathbf{\Sigma}^0_\alpha$.

Proof. $\underline{U \text{ is } \mathbf{\Sigma}^0_\alpha.}$ Indeed

$$U = \bigcup_{n<\omega} V_n \text{ where } V_n = \left\{(x, y) : ((x)_n, y) \in U^{\pi(n)}\right\},$$

which is the preimage of $U^{\pi(n)}$ under the cts map $(x, y) \mapsto ((x)_n, y)$, and $U^{\pi(n)} \in \mathbf{\Pi}^0_{\pi(n)}$, which is boldface, so $V_n \in \mathbf{\Pi}^0_{\pi(n)}$, so $U \in \mathbf{\Sigma}^0_\alpha$.

$\underline{U \text{ is } \omega^\omega\text{-universal for } \mathbf{\Sigma}^0_\alpha.}$ Take $A = \bigcup_{n<\omega} A_n$, where $A_n \in \mathbf{\Pi}^0_{\xi_n}$, $\xi_n < \alpha$.

For each n, find k_n s.t. $\pi(k_n) = \xi_n$ while requiring that k_n are distinct. Find $x_n \in \omega^\omega$ s.t. $\left(U^{\xi_n}\right)_{x_n} = A_n$. For each $\beta < \alpha$, find some z_β s.t. $\left(U^\beta\right)_{z_b} = \varnothing$.

Def x s.t. $(x)_{k_n} = x_n$, and $(x)_l = z_{\pi(l)}$ if $l \neq k_n$ $\forall n$. Now stare at the def of U to see that $U_x = A$. □

□

Remark 3.27. We heavily used AC, every time when we say "*pick*", "*find*", etc. But this is necessary as we have seen that without AC, we could very well have every set $\mathbf{\Sigma}^0_4$, in which case there is no universal $\mathbf{\Sigma}^0_4$-set.

§3.3 The Projective Hierarchy

We already discussed ZFC $\vdash$ "*every Borel set is determined*" , where "discussed" means "*not proved because time is finite*".

Fact 3.28 (Lebesgue's error)**.** He famously claimed cts images of Borel sets are Borel and used this to show that cts images of Borel sets are L-measurable. Indeed they are L-measurable, but the claim was wrong.

> **Theorem 3.29** (Suslin 1917)
>
> The class of all cts images of Borel sets is not closed under complementation. (But the Borel algebra is, so they are not equal). This class is denoted by $\mathfrak{a}$ (or the same letter in some stranger unreproducible font(?))

If $\mathfrak{a}$ is not closed under complement, then let $\check{\mathfrak{a}}$ ("a *dual*") be $\{\omega^\omega \setminus A : A \in \mathfrak{a}\}$ and ask whether it is closed under cts images.

Answer 3.30. NO.

So if $\mathfrak{a}^*$ is the closure of $\check{\mathfrak{a}}$ under cts images, then is it closed under complement? etc.

This is called the **projective hierarchy**. The order type of this hierarchy is just ω because we cannot use transfinite recursion when we only have unary operations available (taking cts images and complementation).

Definition 3.31 (projections)**.** Let $C \subseteq \omega^\omega \times \omega^\omega$. Then def $pC = \{x \in \omega^\omega : \exists y,\ (x, y) \in C\}$ to be the projection into x-axis.

Remark 3.32. It is FALSE that pC is always simpler than C. Indeed if C is in the projective hierarchy, then pC can be one step higher in the hierarchy.

All finite prods $(\omega^\omega)^n$ are homeomorphic to ω^ω: there is a homeomorphism $\omega^\omega \times \omega^\omega \to (\omega \times \omega)^\omega$ because ω^ω has the prod topology of the discrete tplgy on ω, but the discrete tplgy does not care about which set it is on: if $|\mathcal{M}| = \aleph_0$, then $\mathcal{M}^\omega \cong \omega^\omega$. This gives us tree representation lemma in all these spaces:

Lemma 3.33 (tree representation lemma)

$C \subseteq \omega^\omega \times \omega^\omega$ is closed iff $\exists\ T \subseteq (\omega \times \omega)^{<\omega}$ tree s.t. $C = [T]$. The same holds for $(\omega^\omega)^n$.

Definition 3.34. $c : \omega^{<\omega} \to \omega^{<\omega}$ is **coherent** if

(i) $s \subseteq t \implies c(s) \subseteq c(t)$, and

(ii) if $x \in \omega^\omega$, then $|c(x \restriction n)| \to \infty$.

If c is coherent, then def $f_c : \omega^\omega \to \omega^\omega$ by

$$f_c(x) = \bigcup_{n \in \omega} c(x \restriction n).$$

By (i), they agree, and by (ii), the union is infinite, so f_c is well-defined.

Lemma 3.35

f is cts iff $\exists$ coherent c s.t. $f = f_c$.

Proof. Sheet 2. $\square$

Remark 3.36. This gives us a game representation of continuity. Def a game $G(f)$: roughly, I plays x and II have to play $f(x)$ to win. Then f is cts $\iff$ II has win strat. May be in Sheet 2.

Lemma 3.37

Let M be ctble, and $T \subseteq M^{<\omega}$ be a tree with $[T] \neq \varnothing$. There there is a cts surjection $f : \omega^\omega \to [T]$.

Proof. Let $s_\varnothing$ be the first (i.e. closest to the root) splitting node of T. Consider $\{m \in M : s_\varnothing m \in T\}$ which is ctble. So there is a surjection $\pi_\varnothing : \omega \to M_\varnothing$. Recursively do the same on each subtree. $\square$

Definition 3.38. A set $A \subseteq \omega^\omega$ is called **analytic**. If there is $C \subseteq \omega^\omega \times \omega^\omega$ closed s.t. $A = pC = \{y \in \omega^\omega : \exists x,\ (x, y) \in C\}$.

Proposition 3.39

Equivalent are:

(i) A is analytic;

(ii) there is a cts $f : \omega^\omega \to \omega^\omega$ s.t. $A = \operatorname{Im} f$.

Proof. (ii) $\implies$ (i). By the closed graph thm, $f \subseteq \omega^\omega \times \omega^\omega$ is closed, so $\operatorname{Im} f = pf$ is analytic.

(i) $\implies$ (ii). By def, $A = pC$, so we have a tree $T \subseteq (\omega \times \omega)^{<\omega}$ s.t. $A = p[T]$.
By Lemma 3.37, $\exists f : \omega^\omega \to (\omega \times \omega)^\omega$ cts with $\operatorname{Im} f = [T]$, so $A = \operatorname{Im}(p \circ f)$. □

Theorem 3.40

The set of all analytic sets is closed under:

1) ctble unions,

2) ctble intersections,

3) cts images.

Proof. Let X, Y be $(\omega^\omega)^n$ for some $n > 0$.

3) If $f : X \to Y$ is cts and $A \subseteq X$ is analytic, then find $g : \omega^\omega \to X$ with $\operatorname{Im}(g) = A$, then $f \circ g : \omega^\omega \to Y$ is cts and $\operatorname{Im}(f \circ g) = f(A)$.

1) Let $A = \bigcup_{n\in\omega} A_n$, and $f_n : \omega^\omega \twoheadrightarrow A_n$ cts surjection. Then

$$\begin{aligned} f_n^* :\ & \omega^\omega \to nA_n \\ & x \mapsto nf_n(x) \end{aligned}$$

is cts, so nA_n is analytic.

Think of x as $x(0)x^+$. This defines a surjection $\omega^\omega \twoheadrightarrow \bigcup_{n\in\omega} A_n$.

2) Let $A = \bigcap_{n\in\omega} A_n$, where $A_n = pC_n$ where C_n are closed. We define

$$C = \{(x, y) : \forall n,\ ((x)_n, y) \in C_n\}.$$

C closed. $C = \bigcap C_n^*$ where

$$C_n^* = \{(x, y) : ((x)_n, y) \in C_n\}$$

is a cts preimage of C_n, whence closed.

Claim 3.41. $A = \bigcap A_n = pC$.

Proof.

$$\begin{aligned} y \in A \iff & \forall n,\ y \in A_n \\ \iff & \forall n, \exists x_n,\ (x_n, y) \in C_n \\ \iff & \exists x, \forall n,\ ((x)_n, y) \in C_n \qquad \text{by combining } x_n \text{ into } x \\ \iff & \exists x,\ (x, y) \in C \\ \iff & y \in pC. \end{aligned}$$

□

□

Corollary 3.42

Every Borel set is analytic.

Proof. Every closed set is analytic and the closure of $\mathbf{\Pi}^0_1$ under ctble union and intersections is a σ-alg. □

Definition 3.43 (Projective hierarchy)**.** $\mathbf{\Pi}^1_0(X) = \mathbf{\Pi}^0_1(X)$. For n given,

$$\mathbf{\Sigma}^1_{n+1}(X) = \{pC : C \in \mathbf{\Pi}^1_n(\omega^\omega \times X)\},$$

$$\mathbf{\Pi}^1_{n+1}(X) = \{X \setminus A : A \in \mathbf{\Sigma}^1_{n+1}(X)\}.$$

A set is called **projective** if it is in some $\mathbf{\Sigma}^1_n$.

Remark 3.44. Roughly, projections corresponds to an $\exists$ quantifier, and complement corresponds to negation, so together the projective hierarchy corresponds to everything definable in first order logic.

So if we can show that the hierarchy is nice, then we can stay assured that whatever formula we write to describe a set, the set is not too bad.

Theorem 3.45 (Suslin)

The projective hierarchy does not collapse, i.e. $\mathbf{\Sigma}^1_n \neq \mathbf{\Pi}^1_n$.

Proof. By universal sets. We already know that $\mathbf{\Pi}^1_0$ has universal set. If $\mathbf{\Sigma}^1_n$ has, then $\mathbf{\Pi}^1_n$ has. Suppose V is universal for $\mathbf{\Pi}^1_n$.

Construct U universal for $\mathbf{\Sigma}^1_{n+1}$. We shall construct these for all finite dim, as the universal set for $\mathbf{\Sigma}^1_{n+1}$ in $(\omega^\omega)^k$ requires a $\mathbf{\Pi}^1_n$-set in $(\omega^\omega)^{k+1}$. Suppose V in ω^ω-universal for $\mathbf{\Pi}^1_n((\omega^\omega)^{k+1})$. Let

$$U = \{(x, x_1, \cdots, x_k) \in \omega^\omega \times (\omega^\omega)^k : \exists z \text{ s.t. } (x, z, x_1, \cdots, x_k) \in V\}.$$

Then U is $\mathbf{\Sigma}^1_{n+1}$. because we are projecting along the second coord, but flipping the first two coords is a homeomorphism.

Let $A \subseteq (\omega^\omega)^k$ be arbitrary $\mathbf{\Sigma}^1_{n+1}$. Then $A = pC$, for some C that is $\mathbf{\Pi}^1_n((\omega^\omega)^{k+1})$. So $C = V_x$ for some x by universality of V. So $A = pV_x = U_x$. □

Remark 3.46. If you close the Borel sets under $\exists x \in \omega^\omega$ and $\forall x \in \omega^\omega$ and propositional connectives $\implies, \perp$, then each set obtained is projective.

§3.4 Regularity Properties

Question 3.47. Are all projective sets nice?

We shall discuss three notions of niceness, aka "regularity properties".

§3.4.1 Lebesgue Measurability

Usually on $\mathbb{R}$, not ω^ω, but we can assign measure $\frac{1}{2^n}$ to $[p]$ with $|p| = n$ and extend by Carathéodory extension thm to the smallest σ-alg containing the basic open sets (=Borel sets). This is called **Borel measurabiltiy**.

If $A \subseteq \omega^\omega$ is arbitrary, then we def $\sup\{\mu(B) : B \subseteq A, B \text{ Borel}\}$ to be the **inner measure**, and $\inf\{\mu(B) : B \supseteq A, B \text{ Borel}\}$ to be the **outer measure**, and we call A **Lebesgue measurable** if these are equal.

Equivalently, A is Lebesgue measurable iff $\exists B$ Borel with $B \subseteq A$ and $\mu(B) = \mu(A)$, i.e. $A \setminus B$ is Lebesgue null (becasue if (B_n) is a sequence tending to the inner measure then $\bigcup B_n$ is Borel so the sup is attained).

§3.4.2 Baire Property/Property of Baire

Let X be an arbitrary topological space. Say $A \subseteq X$ is **nowhere dense** if there is no open set s.t. A is dense in U, i.e. $\mathrm{Int}(\mathrm{Cl}(A)) = \varnothing$. This should be thought as small.

A set is **meagre** if it is ctble union of nowhere dense sets. These are also small but can already be dense.

Example 3.48

If X is metric, then $\{x\}$ is nowhere dense, so every ctble set is meagre, e.g. $\mathbb{Q} \subseteq \mathbb{R}$ is meagre but dense.

Theorem 3.49 (Baire category thm)

In $\mathbb{R}$, no open set is meagre.

Definition 3.50. If X satisfies BCT, then X is **Baire**.

Example 3.51

THE Baire space ω^ω is A Baire space.

Remark 3.52. If you become famous, then make sure things named after you are compatible.

Definition 3.53. $A \subseteq X$ has the **Baire property** ($\neq$ being Baire) if there is a Borel set B s.t. the symmetric difference $A \bigtriangleup B$ is meagre.

Remark 3.54. If we have a def of smallness, then we can say sth is regular if it is nice except for some small portion.

§3.4.3 The Perfect Set Property (PSP)

Definition 3.55. $P \subseteq X$ is **perfect** if P is closed and has no isolated points.

Theorem 3.56 (Cantor–Bendixson)

If $F \subseteq \mathbb{R}$ closed, then either F is ctble or $F \supseteq P$ for some $P \neq \varnothing$ perfect.

Proof. Very old application of transfinite recursion by removing isolated points. If we removed everything, then we are in the first case. O/w we are in the second case at some point. □

If $F \subseteq \omega^\omega$ is perfect, then F is closed, so $F = [T_F]$ by tree rep thm. An isolated point in $[T_F]$ means there is a node on a branch s.t. the branch never splits after that node.

Definition 3.57. We call a tree **perfect** if $\forall t \in T,\ \exists s \supseteq t$ s.t. $s \in T$ and s splits in T.

We observe:

1. F is perfect $\iff$ T_F is perfect. (Also holds for $F = \varnothing$) Strat trees are perfect since branches split at every second level.
2. If $T \neq \varnothing$ is perfect, then $|[T]| = 2^{\aleph_0}$.
3. This means we can phrase Cantor–Bendixson as: No closed sets can be counterexample to the Continuum Hypothesis CH.
4. There were attempts to show CB for all sets in order to show CH, leading to the following def.

Definition 3.58. A has **perfect set property (PSP)** if either A is ctble, or A includes a non-empty perfect set.

Remark 3.59. Not as good as previous niceness but it is like determinacy that A can be PSP by being nice very locally but really nasty elsewhere.

Think about our proof that $\mathsf{AC} \implies \exists$ non-determined set. The same proves $\mathsf{AC} \implies \exists$ set without PSP (list all PSPs, diag-argue...).

Every projective set is regular where regular means one of the above (LM/BP/PSP). (There is a tplgy on ω^ω s.t. BP in this tplgy is something called **Ramsey property**.)

We have seen the sketch that $\mathsf{AC} \implies \exists$ set without PSP. If AC is "*too definable*", then this might result in a projective set without PSP. (o/w the choice function is mixing up elements of different complexity). Note that AC is equivalent to the well-ordering principle, and we only used AC restricted to $\mathcal{P}(\omega^\omega) \setminus \{\varnothing\}$: a choice function for this set is enough for all proofs. But this is equivalent to "there is a well-ordering of ω^ω", i.e. some $R \subseteq \omega^\omega \times \omega^\omega$ that is irreflexive, transitive, and wellfounded.

It therefore makes sense to ask whether the well-ordering as a subset is Borel/projective/....

Sheet 2 says a well-ordering of ω^ω means that not every proj set is regular. Thus, if R is a proj wellordering of ω^ω, then the proof of $\mathsf{AC} \implies \neg\mathsf{PSP}$ gives a proj non-PSP set. When we constructed the set, we can use instead

$$\begin{aligned} a_\alpha &= \min_R [T_\alpha] \setminus (A_\alpha \cup B_\alpha), \\ b_\alpha &= \min_R [T_\alpha] \setminus (A_\alpha \cup B_\alpha \cup \{a_\alpha\}. \end{aligned}$$

instead of AC.

Exercise 3.60. Express $A = \bigcup_{\alpha < 2^{\aleph_0}} A_\alpha$ as proj set if R is proj.

Proposition 3.61

AC $\implies \aleph_1$ regular.

Proof. Encode ctble ordinals as elements of ω^ω as follows: We split $x \in \omega^\omega$ into x_{I}, x_{II}, and def $A_x = \{u : x_{\mathrm{I}}(u) \neq 0\} \subseteq \omega$. Fix $b : \omega \times \omega \to \omega$ bijection, then let

$$R_x = \{(n, m) \in A_x^2 : x_{\mathrm{II}}(b(n, m)) \neq 0\} \subseteq A_x \times A_x.$$

We map x to

$$\|x\| = \begin{cases} \alpha, & \text{if } (A_x, R_x) \cong (\alpha, \in), \alpha \text{ ctble ordinal,} \\ 0, & \text{o/w.} \end{cases}$$

Note that $x \mapsto \|x\|$, $\omega^\omega \to \aleph_1$ is a surjection in ZF. □

§3.5 WO and WO$_\alpha$

Definition 3.62.

$$\mathrm{WO}_\alpha = \{x : \|x\| = \alpha\}, \ \mathrm{WO} = \bigcup_{\alpha < \aleph_1} \mathrm{WO}_\alpha.$$

WO is the set of codes of ctble ordinals.

Analysing the proof of AC $\implies \aleph_1$ regular, we observe that we only need a choice function for $\{\mathrm{WO}_\alpha : \alpha < \aleph_1\}$.

Fact 3.63. The Feferman–Lévy model is a model of ZF+"$\aleph_1$ *is not regular*", so we cannot pick from the set $\{\mathrm{WO}_\alpha : \alpha < \aleph_1\}$ is that model.

What if we have a family of closed sets? If $C \subseteq \omega^\omega$ is closed, then $C = [T_C]$. If $C \neq \varnothing$, then $[T_c] \neq \varnothing$. Pick recursively the leftmost branch by always picking the least n s.t. there is a branch extending the place where we are currently at. Note that we used that it is a tree on ω which is well-ordered.

If Z is a family of nonempty closed sets, then $C \mapsto$ (leftmost branch of T_C) is a choice for Z.

This implies not all WO$_\alpha$ are closed (really crude lower bound on their complexity). If they were, then the above argument (in ZF proves $\aleph_1$ regular, contradicting Feferman–Lévy.

Question 3.64. What is the complexity of WO$_\alpha$?

Approximately Σ^0_α.

Question 3.65. What is the complexity of WO?

This is $\{x : (A_x, R_x)$ is a well order$\}$ which can be written as $W_1 \cap W_2 \cap W_3$, where

$$\begin{aligned} W_1 &= \{x : (A_x, R_x) \text{ is irreflexive}\}, \\ W_2 &= \{x : (A_x, R_x) \text{ is transitive}\}, \\ W_3 &= \{x : (A_x, R_x) \text{ is wellfdd}\}. \end{aligned}$$

We have

$$W_1 = \{x : \forall n, \quad \underbrace{x_{\mathrm{I}}(n) \neq 0 \implies x_{\mathrm{II}}(b(n, n)) = 0}_{\substack{\text{this set } W_{1n} \text{ is clopen because it only depends} \\ \text{on two points in the sequence}}} \quad \},$$

so $W_1 = \bigcap_{n\in\omega} W_{1n}$ is closed ($\mathbf{\Pi}^0_1$).

$$\begin{aligned} W_2 = \{x : \forall n\forall m\forall k,\ (&x_{\mathrm{I}}(n) \neq 0 \wedge x_{\mathrm{I}}(m) \neq 0 \wedge x_{\mathrm{I}}(k) \neq 0 \\ &\wedge x_{\mathrm{II}}(b(n,m)) \neq 0 \wedge x_{\mathrm{II}}(b(m,k)) \neq 0) \implies x_{\mathrm{II}}(b(n,k)) \neq 0\}. \end{aligned}$$

For fixed n, m, k, the set W_{2nmk} is clopen, so W_2 is closed. We should get nervous because if W_3 is also closed, then WO is closed and set theory is inconsistent and we can go home (X).

Definition 3.66. A relation is **illfounded** if it is not wellfdd. In this case we have a sequence $(a_n)_{n\in\omega}$ s.t. $a_{n+1} R a_n$.

We can write $W_3 = \omega^\omega \setminus IF$, where

$$IF = \{x : \exists y \in \omega^\omega,\ \forall k,\ x_{\mathrm{I}}(y(k)) \neq 0 \wedge \forall k,\ x_{\mathrm{II}}(b(y(k+1), y(k))) \neq 0\}.$$

Let

$$C = \{(x,y) : \forall k,\ x_{\mathrm{I}}(y(k)) \neq 0 \wedge x_{\mathrm{II}}(b(y(k+1), y(k))) \neq 0\}.$$

Then by the same argument, C is closed in $\omega^\omega \times \omega^\omega$. So $IF = pC$ is $\mathbf{\Sigma}^1_1$, so W_3 is $\mathbf{\Pi}^1_1$.

Hence $\mathrm{WO} = W_1 \cap W_2 \cap W_3$ is $\mathbf{\Pi}^1_1$. If A is $\mathbf{\Sigma}^1_1$, then there is a closed set C s.t.

$$x \in A \iff \exists y \text{ s.t. } (y,x) \in C,$$

so by tree rep, we have some tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ s.t.

$$\begin{aligned} x \in A &\iff \exists y,\ (y,x) \in [T] \\ &\iff \exists y,\ \forall n,\ (y\restriction n, x\restriction n) \in T. \end{aligned}$$

Def $T_x = \{t : (t, x\restriction |t|) \in T\}$. This is a tree $\subseteq \omega^{<\omega}$. Then

$$x \in A \iff \exists y,\ \forall n,\ y\restriction n \in T_x \iff \exists y,\ y \in [T_x] \iff [T_x] \neq \varnothing \iff T_x \text{ is illfdd.}$$

Going to the complement, we have

Proposition 3.67 (The tree representation for $\mathbf{\Pi}^1_1$ sets)
$A \subseteq \omega^\omega$ is $\mathbf{\Pi}^1_1$ iff there is a tree T on $\omega \times \omega$ s.t. $x \in A \iff T_x$ is wellfdd.

Remark 3.68. WO is a particular example which comes with a "natural stratification" in $\aleph_1$ many layers, and the tree representation allows us to do this for arbitrary $\mathbf{\Pi}^1_1$ sets.

If A is $\mathbf{\Pi}^1_1$ and T its tree rep, then we write

$$A_\alpha = \{x : \mathrm{ht}(T_x) \leq \alpha\} \subseteq A.$$

We have $A = \bigcup_{\alpha<\aleph_1} A_\alpha$.

Lemma 3.69
A_α are Borel.

Proof. Induction.

$$\begin{aligned} A_0 &= \{x : \mathrm{ht}(T_x) = 0\} \\ &= \{x : \forall i,\ (i) \notin T_x\} \\ &= \{x : \forall i,\ (i, x(0)) \notin T\} \end{aligned}$$

which is an intersection of open sets, whence $\mathbf{\Pi}^0_2$.

If $s \in \omega^{<\omega}$ and T is a tree on ω, then we def the tree "T *from* s" by $T|s = \{t : st \in T\}$. Let $A^s_\alpha = \{x : \mathrm{ht}(T_x|s) \leq \alpha\}$. Same argument as before, $A^s_0 \in \mathbf{\Pi}^0_2 \quad \forall s$. Assume α given and A^s_β Borel for all $\beta < \alpha$.

Note that

$$A^s_\alpha = \bigcap_{i\in\omega} \bigcup_{\beta<\alpha} A^{si}_\beta$$

. (height at most α means all successors have height less than α). □

Therefore we have shown the classical theorem:

Theorem 3.70

Every $\mathbf{\Pi}^1_1$ set is a union of $\aleph_1$ many Borel sets.

Instead of WO, now we look at $\mathrm{WF} = \{x : (A_x, R_x) \text{ is wellfdd}\}$, and $\mathrm{WF}_\alpha = \{x : \mathrm{ht}(A_x, R_x) \leq \alpha\}$.

For WF, we seet that the stratification given by the proof corresponds to WF_α.

Remark 3.71. We can analyse WO like this by linearising the trees in A_α by Kleene–Brouwer ordering on Sheet 3.

Theorem 3.72 (Boundedness)

Let $A \subseteq \mathrm{WF}$ be $\mathbf{\Sigma}^1_1$, then there is an $\alpha < \aleph_1$ s.t. $A \subseteq \mathrm{WF}_\alpha$, i.e. $\mathbf{\Sigma}^1_1$ sets are bounded.

Proof. Assume A is $\mathbf{\Sigma}^1_1$ and $A \subseteq \mathrm{WF}$, and $\forall\alpha,\ A \not\subseteq \mathrm{WF}_\alpha$. We shall show that every $\mathbf{\Pi}^1_1$ set is $\mathbf{\Sigma}^1_1$, thus contradicting the hierarchy thm.

Let P be an arbitrary $\mathbf{\Pi}^1_1$ set. By tree representation, get T tree s.t. $x \in P \iff T_x$ wellfdd. Since WF is also $\mathbf{\Pi}^1_1$, write U for its tree, so that $y \in \mathrm{WF} \iff U_y$ wellfdd. Since $x \in P \iff T_x$ wellfdd, we can find α s.t. $\mathrm{ht}(T_x) \leq \alpha$, and now find $y \in A$ s.t. $\mathrm{ht}(U_y) > \alpha$ by unboundedness of A, so that there is an order-preserving map $T_x \to U_y$, so

$$x \in P \iff \exists y \text{ s.t. } (y \in A \wedge \text{ there is an order-preserving map } T_x \to U_y).$$

But A is $\mathbf{\Sigma}^1_1$, and the existence of order-preserving map is a low level Borel property, so P is $\mathbf{\Sigma}^1_1$, contradiction. □

Corollary 3.73

If $X \subseteq \mathrm{WF}$ s.t. for each $\alpha < \aleph_1$ there is precisely one $x \in X$ s.t. $\mathrm{ht}(U_x) = \alpha$ (i.e. some sort of choice function), then X cannot have the perfect set property.

Proof. $|x| = \aleph_1$, so if it has PSP, then it contains a non-empty perfect set, but it is closed, so in particular $\boldsymbol{\Sigma}^1_1$. By bonndedness, it is bounded, but bounded subsets of X are ctble, contradiction. $\square$

Corollary 3.74
AC $\implies \neg$PSP.

Proof. If there is a projective well order of ω^ω, then there is a projective set without PSP. $\square$

§3.6 Back to Games

The goal is to show the opposite: If all projective sets are determined, then all projective sets have PSP.

Definition 3.75 (Perfect games)**.** Let $A \subseteq 2^\omega$ (can be done on ω^ω but 2^ω is more convenient). We def $G^*(A)$ to be the game with game play

$$\begin{array}{lcccccccc} \mathrm{I} & s_0 \in 2^{<\omega} & & s_1 \in 2^{<\omega} & & s_2 & & \\ \mathrm{II} & & x_0 \in 2 & & x_1 \in 2 & & x_2 & \cdots \end{array}$$

Def $x^* = s_0 x_0 s_1 x_1 \cdots$. I wins if $x^* \in A$. Note that $G^*(A)$ is the same game as $G(A^*)$ where $b : \omega \to 2^{<\omega}$ is a bijection, $m : \omega \to 2,\ n \mapsto n \mod 2$, and

$$A^* = \{x : b(x(0))m(x(1))b(x(2))m(x(3)) \cdots \in A\}.$$

So A^* is a cts preimage of A, i.e. we have a map

$$(2^{<\omega} \times 2)^{<\omega} \to 2^{<\omega},\ z \mapsto z^*.$$

This shows:

Proposition 3.76
If $\boldsymbol{\Gamma}$ is boldface pointclass and all games in $\boldsymbol{\Gamma}$ are determined, then all games $G^*(A)$ are determined for $A \in \boldsymbol{\Gamma}$.

Theorem 3.77
Let $A \subseteq 2^\omega$. Then

1) If I has win strat in $G^*(A)$, then A contains a non-empty perfect set.

2) If II has win strat in $G^*(A)$, then A is ctble.

Proof. 1) If σ is a win strat for I, then the strat tree gives a perfect subtree T of $2^{<\omega}$: the tree splits below every node because II can choose to play 0 or 1.

Now $[T] \subseteq A$ since σ is winning. (The converse is also true: If $[T] \subseteq A$ and T perfect then we can read off a win strat for I).

2) Let τ be a win strat for II, then $(\sigma * \tau)^* \notin A$, so if $z \in A$, then there is no way writing $z = x^*$ where $x \in (2^{<\omega} \times 2)^{<\omega}$, and x follows the strat τ.

If $p \in (2^{<\omega} \times 2)^{<\omega}$ is a position, then we say that p is τ-consistent if $\forall n \in |p|$, we have that if $p(n) = (s, x)$, then $x = \tau(p \upharpoonright n, s)$. Hence if $z \in A$, then there is a maximal τ-consistent position p s.t. $p^* \subseteq z$ (o/w we can def a τ-play by recursion to obtain $z \notin A$.)

If p is max, then any extension q is either not τ-consistent or $q^* \not\subseteq z$. So if the next move of I is $\varnothing$, then τ must respond by $1 - z(k)$, k some move number. If I plays $z(k)$ instead, then τ responds by $1 - z(k+1)$, and so on.

This says if p is a max position, then p uniquely determines z by recursion, giving a surjection from $(2^{<\omega} \times 2)^{<\omega}$ onto A, so A is ctble. □

Remark 3.78. This proof does not work if we replace 2 by 3.

§4 Large Cardinals

We have seen propositions such as PWO "*There is a projective well-ordering of ω^ω*", PPSP "*All projective sets have PSP*", and PD "*Projective determinacy*". In fact PD $\implies$ PPSP $\implies$ $\neg$PWO. These are related to large cardinal axioms, and are *interesting* (i.e. indep of ZFC).

Fact 4.1 (History)**.** In the 60's, it was known that there is a close connection between large cardinals and proj regularity. Solovay has shown that if IC "*there is an inaccessible cardinal*", then ProjLM (L-measurable) is consistent. At the time, it was open whether the IC is needed (Shelah 1984: It is).

So it was clear to the set theorists of the 60's that PD should have strength beyond ZFC.

Definition 4.2. If β is an ordinal, then we call $C \subseteq \beta$ **cofinal** or **unbounded** if $\forall \alpha < \beta$, $\exists \gamma \in C$ s.t. $\gamma > \alpha$. This is mostly of interest for limits (o/w there is a max element).

Definition 4.3. The **cofinality** $\operatorname{cf}(\lambda) = \min\{|C| : C \subseteq \lambda \text{ cofinal}\}$. Clearly $\operatorname{cf}(\lambda) \leq |\lambda|$. A cardinal κ is **regular** if $\operatorname{cf}(\kappa) = \kappa$, **singular** if $\operatorname{cf}(\kappa) < \kappa$.

With AC, we know that successor cardinals are always regular, e.g. $\aleph_1 = \aleph_0^+$ on Sheet 1.

Limit cardinals often are not: $\{\aleph_n : n \in \omega\} \subseteq \aleph_\omega$ is cofinal, so $\operatorname{cf}(\aleph_\omega) = \aleph_0$, so $\aleph_\omega$ is singular. Simly, $\aleph_{\aleph_1}$ has cf $\aleph_1$.

Question 4.4. Are there regular limits?

Observe that if λ is a limit ordinal, then $\operatorname{cf}\aleph_\lambda = \operatorname{cf}\lambda$, so a necessary condition for κ to be a regular limit is that $\aleph_\kappa = \kappa$. Normal functions have a lot of fixed points, e.g. $\sup\{\aleph_0, \aleph_{\aleph_0}, \cdots\}$, but this only has cf ω, so it is still tiny compared to regular limits.

Definition 4.5. κ is **weakly inaccessible** if it is a reg limit cardinal. κ is **strongly inaccessible** if it is a reg **strong limit** cardinal, where strong limit says $\forall \lambda < \kappa$, $2^\lambda < \kappa$.

These def are by Hausdorff.

We can show ZFC $\nvdash \exists$ strongly inaccessible cardinals. (also ZFC $\nvdash \exists$ weakly inaccessible cardinals but this is difficult)

§4.1 Basic Model Theory of Set Theory

This is done by the Lecturer's Part III Topics in Set Theory in L2019. Also check Kunen, *Set Theory*, Ch 4 (Easy Consistency Proofs) Sec 3 (Absoluteness).

We work within a fixed model $(V, \in) \models \mathsf{ZFC}$. Consider models **transitive** in V. $M \subseteq V$ is transitive if $m \in M$ and $x \in m$ implies $x \in M$.

If $(M, \in)$ is considered as a structure, then many definitions retain their meaning between V and M, i.e. many formulae are absolute: if they hold in V, then also in M, and vice versa.

Example 4.6

1. If $M \models x = \mathbb{N}$ (i.e. x is the unique object that is subset to all inductive sets), then $V \models x = \mathbb{N}$.
2. If $M \models f : x \to y$, then $V \models f : x \to y$. The same holds for injectivity, surjectivity, whence bijectivity, and the reverse direction also works if $x, y, f \in M$.
3. If $M \models x \subseteq y$, then $V \models x \subseteq y$.
4. However, $M \models x = \mathcal{P}(y)$ does not mean $V \models x = \mathcal{P}(y)$ because y might have more subsets in V.
5. In general, existential formulae are upward preserved.
6. If $(M, \in) \subseteq (V, \in)$ and φ is a **quantifier-free** formula, then $(M, \in) \models \varphi \iff (V, \in) \models \varphi$. But most things are not qff when we spell it out, e.g. $x = \varnothing$ says $\forall y,\ y \notin x$.

Consider the **von Neumann hierarchy** defined by

$$V_0 = \varnothing, \quad V_{\alpha+1} = \mathcal{P}(V_\alpha), \quad V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha.$$

Exercise 4.7 (Tripos questions sometimes)**.**

1. V_α are transitive.
2. If $\lambda > \omega$ is a limit, then $V_\lambda \models \mathsf{ZC} + \mathsf{Foundation}$, where $\mathsf{Replacement}$ is due to Fraekel so we have left out F. To spell these out: Empty, $\mathsf{Extension}$, $\mathsf{Pairing}$, Union, $\mathsf{Separation}$, Choice, $\mathsf{Powerset}$, $\mathsf{Infinity}$.

Property	Implication (true in $M \overset{?}{\iff}$ true in V) Special case when $M = V_\lambda$, λ limit
$f : x \to y$ f injective, surjective, bijective	$\iff$ (**absolute**) $\iff$
α is an ordinal	$\iff$ (only non-trivial thing is wellfddness, but any infinite descending seq in V is in M by transitivity)
$x = \mathbb{N}$ (n.b. not quantifier-free)	$\iff$
x is ctble (this sentence is Σ_1)	$\implies$ (**upward absolute**). Intuitively, M may not see the surjection $f : \mathbb{N} \to x$ in V, so we expect $\not\Longleftarrow$ (this is not a proof)

Property	Implication (true in $M \overset{?}{\iff}$ true in V) Special case when $M = V_\lambda$, λ limit
α is a cardinal, i.e. smaller ordinals do not surject to α. (Π_1)	$\Longleftarrow$ (**downward absolute**). $\iff$. If α not card, then the surjectiion $f : \beta \to \alpha$ satisfies $$f \in \mathcal{P}(\beta \times \alpha) \subseteq V_{\alpha+\beta} \subseteq V_\lambda.$$
κ is regular, i.e. smaller sets are not cofinal	$\Longleftarrow$. $\iff$. The cofinal subset is also in V_λ.
$x \subseteq y$	$\iff$ (transitivity is key)
$x = \mathcal{P}(y)$	$\Longleftarrow$. y can have more subsets in V. $\iff$ since V_λ is defined by power sets.
κ is a limit cardinal	$\Longleftarrow$
κ is weakly inacc	$\Longleftarrow$ (we have seen regularity and limit)
κ is a strong limit	$\Longleftarrow$ (if $M \models 2^\lambda \geq \kappa$, then M has $f : \mathcal{P}(\lambda) \twoheadrightarrow \kappa$, but that in V is a surjection from a subset of $\mathcal{P}(\lambda)$ to κ.)
κ is strongly inacc	$\Longleftarrow$ by previous

For models of the form V_λ, λ limit, we have

$$V \models \kappa \text{ strongly inacc} \iff V_\lambda \models \kappa \text{ strongly inacc}.$$

The same holds for weakly.

We shall prove the following:

Theorem 4.8
If κ is strongly inacc, then $V_\kappa \models \mathsf{ZFC}$.

Corollary 4.9
$\mathsf{ZFC} \nvdash \mathsf{IC}$ "*there is a strongly inacc card*", assuming that ZFC is consistent.

Proof 1 of Cor. If $\mathsf{ZFC} \vdash \mathsf{IC}$, then $\mathsf{ZFC} \vdash \exists \kappa\ (V_\kappa \models \mathsf{ZFC})$ (note that this says $\forall n \in \omega$, the n-th item in our list of propositions is a thm ...). By completeness, $\mathsf{ZFC} \vdash \mathsf{Cons}(\mathsf{ZFC})$, so by Gödel's incompleteness, ZFC is not consistent. □

Proof 2 of Cor. Suppose $\mathsf{ZFC} \vdash \mathsf{IC}$. Work in some model V of ZFC (n.b. we are using ZFC consistent.) Let $\kappa_0 \in V$ be the least inacc cardinal. Look at V_{κ_0}.

This is a model of ZFC, so $V_{\kappa_0} \models \mathsf{ZFC} + \mathsf{IC}$. Find $\lambda < \kappa_0$ s.t. $V_{\kappa_0} \models \lambda$ inacc, so $V \models \lambda$ inacc, contradicting minimality of κ_0. □

Now it suffices to prove $V_\kappa \models \mathsf{Replacement}$. We shall show sth stronger, i.e. if $F : V_\kappa \to V_\kappa$, and $x \in V_\kappa$, then $F[x] \in V_\kappa$. ($\mathsf{Replacement}$ only says this for definable F)

"*You can avoid all the logic by just counting things*"

Lemma 4.10

If κ strongly inacc and $\lambda < \kappa$, then $|V_\lambda| < \kappa$.

Proof. *"Blatantly obvious"*. Induction. $|V_\omega| = \aleph_0 < \kappa$. Suppose $|V_\alpha| < \kappa$, then $|V_{\alpha+1}| = |\mathcal{P}(V_\alpha)| = 2^{|V_\alpha|} < \kappa$ since κ is strong limit.

If $\lambda < \kappa$ is a limit cardinal, then $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$, so

$$|V_\lambda| \leq \sum_{\alpha<\lambda} |V_\alpha| < \kappa$$

by regularity. □

Lemma 4.11

Suppose κ is strongly inacc. Take some $A \subseteq \kappa$. Then $A \in V_\kappa \iff |A| < \kappa$.

Proof. $\underline{\implies}$ If $A \in V_\kappa = \bigcup_{\lambda<\kappa} V_\lambda$, then $\exists \lambda < \kappa$ s.t. $A \in V_\lambda$. Since V_λ is transitive, $A \subseteq V_\lambda$, so $|A| \leq |V_\lambda| < \kappa$ by Lemma 4.10.

$\underline{\impliedby}$ $A \subseteq V_\kappa$, $|A| < \kappa$, so $\forall a \in A, \exists \alpha < \kappa$ s.t. $a \in V_{\alpha+1} \setminus V_\alpha$ (at lim levels nothing new is added). Write $\rho(a) = \alpha$. (Called the **Mirimanoff rank**).

Consider

$$A^* = \{\rho(a) + 1 : a \in A\} \subseteq \kappa,$$

then $|A^*| \leq |A| < \kappa$. By rglrty, A^* is bounded by some $\lambda < \kappa$, so $A \subseteq V_\lambda$, so $A \in \mathcal{P}(V_\lambda) = V_{\lambda+1} \subseteq V_\kappa$. *"Very elementary."* □

Proof of Theorem 4.8. Let $F : V_\kappa \to V_\kappa$, and $x \in V_\kappa$. By Lemma 4.11, $|x| < \kappa$, so $|F[x]| \leq |x| < \kappa$. But $F[x] \subseteq V_\kappa$, and $|F[x]| < \kappa$, so by Lemma 4.11 again, $F[x] \in V_\kappa$. □

Remark 4.12. $V_{\omega+\omega}$ goes wrong because ω is bounded but we can map it to $\{\omega+1, \omega+2, \cdots\}$ which is unbounded in $V_{\omega+\omega}$, so $V_{\omega+\omega} \not\models \mathsf{Replacement}$.

$V_{\aleph_1}$ does not work because $\mathcal{P}(\omega) \in V_{\omega+2}$ but surjects to $\aleph_1$.

§4.2 Real Model Families

Definition 4.13. We call a familty $\{M_x : x \in \mathbb{R}\}$ a **real model family** if each $M_x \models \mathsf{ZFC}$ and is transitive, and for each x, $x \in M_x$, $M_0 \subseteq M_x$, and $\mathrm{On}^{M_x} = \mathrm{On}^V$ (i.e. they have the same ordinals), where 0 is the real number 0 in your favourite construction for $\mathbb{R}$.

"Did I tell you I don't like the real numbers?"

Note that we can also use ω^ω instead of $\mathbb{R}$ as they biject. We also write $M = M_0$.

Remark 4.14. This is non-standard term.

Example 4.15

If $\mathbb{R} \subseteq M_0$, then $M_x = M_0$ is a real model family.

Definition 4.16. An RMF **satisfies** CH if each model satisfies CH. An RMF is **projectively well-ordered** if for each x, the set $\omega^\omega \cap M_x$ has a projective well-ordering (in V).

Definition 4.17. We say $\aleph_1$ **is inaccessible by reals for** $\overrightarrow{M}$, $\overrightarrow{M}$ RMF, if $\forall x$, $\aleph_1^{M_x} < \aleph_1^V$, i.e. M_x thinks there is a smaller unctble cardinal. "*They are wrong everywhere*".

Lemma 4.18

If $\overrightarrow{M}$ is an RMF, s.t. $\aleph_1$ is inacc by reals for it, then $M = M_0 \models$ "$\aleph_1^V$ is a weakly inacc cardinal".

If moreover $\overrightarrow{M}$ satisfies CH, then $M \models$ "$\aleph_1^V$ is strongly inacc".

Proof. Need to show that $\aleph_1^V$ is a regular and limit cardinal. Since $\aleph_1$ is reg in V, and rglrty is downward absolute, $M \models$ "$\aleph_1^V$ regular".

Suppose $\aleph_1^V$ is not lim in M, then there is $\kappa < \aleph_1^V$, s.t. $M \models \aleph_1^V = \kappa^+$, i.e. no cardinals between $\aleph_1^V$ and κ.

Since $\kappa < \aleph_1^V$, there is some $x \in \omega^\omega$ s.t. $(A_x, R_x) \cong (\kappa, \in)$, i.e. $x \in \mathrm{WO}_\kappa$. Clearly, if $x \in N$, then $N \models$ "x is ctble", so go to $M_x \supseteq M_0$, and we have $M_x \models$ "κ is ctble".

Since being a cardinal is downward absolute, in M_x we have $\aleph_1^V$ is the least unctble cardinal, so $\aleph_1^V = \aleph_1^{M_x}$, contradiction.

Moreover, suppose there is no $\kappa < \aleph_1^V$ s.t. $M \models 2^\kappa \geq \aleph_1^V$, i.e. there is $f \in M$ s.t.

$$f : P = \mathcal{P}(\kappa) \cap M \to \aleph_1^V \text{ surjective.}$$

Since $\kappa < \aleph_1^V$, find x s.t. $x \in \mathrm{WO}_\kappa$. Then in M_x, we find $P \in M \subseteq M_x$, $f \in M \subseteq M_x$, and $x \in M_x$, so using the bijection between ω and κ from x, we put the above together to show $P' \subseteq \mathcal{P}(\omega)$, where P' is the image of P under the inverse of the bijection, and $f' : P' \to \aleph_1^V$ surjection. Hence $M_x \models 2^{\aleph_0} \geq \aleph_1^V$.

If moreover $M_x \models \mathsf{CH}$, then $M_x \models \aleph_1^{M_x} \geq \aleph_1^V$, contradicting $\aleph_1^V$ inacc by reals. $\square$

Definition 4.19. $A \subseteq \mathrm{WO}$ is called a **set of unique codes** if $\forall x \in \mathrm{WO}$, there is at most one $y \in A$ s.t. $\|x\| = \|y\|$. (Recall $x \in \mathrm{WO}$ says $(A_x, R_x) \cong (\alpha, \in)$ and $\|x\| = \alpha$.) SUCs are thin, i.e. they intersect each layer of WO at most once.

We have already shown the following:

Lemma 4.20

An SUC cannot contain a non-empty perfect set. In particular if an SUC has PSP, then it is ctble.

Proof. If $P \subseteq A$ perfect, then P closed, so $\boldsymbol{\Sigma}^1_1$, so by boundedness, $P \subseteq \mathrm{WO}_{\leq \alpha}$, $\alpha < \aleph_1$, so $|A \cap \mathrm{WO}_\alpha| \leq |\alpha|$, contradicting P unctble. $\square$

Lemma 4.21

If there is a proj well-ordering of $S \subseteq \omega^\omega$, then there is a proj SUC A s.t. $\forall x \in S \cap \mathrm{WO}$, $\exists y \in A$ s.t. $\|x\| = \|y\|$.

Theorem 4.22

If $\overrightarrow{M}$ is a projly wellorded RMF, and every proj set has PSP, then $\aleph_1$ is inacc by reals for $\overrightarrow{M}$.

Proof. Since $\overrightarrow{M}$ is projly wellorded, we have proj wellorders of $\omega^\omega \cap M_x\ \forall x$. By Lemma 4.21, there is an SUC A_x s.t.

$$\forall z \in \mathrm{WO} \cap M_x,\ \exists y \in A_x \text{ s.t. } \|y\| = \|z\|. \tag{$*$}$$

Assume to the contrary that we have some x s.t. $\aleph_1^{M_x} = \aleph_1^V$. By $(*)$, this means $|A_x| = \aleph_1$. A_x is projective, so A_x has PSP, but now we have an unctble SUC with PSP, contradicting Lemma 4.20. □

Theorem 4.23 (Gödel 1938)

There is an RMF which is projly wellorded and satisfies CH. This is called the **constructible universe** $\boldsymbol{L}$.

We shall use this as a black box.

Remark 4.24. This can be thought of as the smallest model of set theory. It is the "closure" under the axioms, but very much more complicated, since when we use $\mathsf{Separation}$, the meanings of the formulae changes after we add new things to the universe.

Corollary 4.25

If all proj sets have PSP, then there is a model of $\mathsf{ZFC} + \mathsf{IC}$.

Corollary 4.26

$\mathsf{ZFC} \not\models$ "all proj sets have PSP".

Corollary 4.27

$\mathsf{ZFC} \not\models \mathsf{PD}$.

Proof. We have seen $\mathsf{PD} \implies \mathsf{PPSP}$. □

If you think ZFC is everything, then we have fully settled determinacy, except ...

Question 4.28. What about determinacy for $\boldsymbol{\Sigma}^1_n$ for fixed n?

Answer 4.29. The Borel sets form the largest class for which ZFC proves determinacy, because $\mathsf{Det}(\boldsymbol{\Sigma}^1_1)$ implies large cardinals.

Question 4.30. Can we get inverse thms e.g. $\mathsf{IC} \implies \mathsf{PD}$?

Answer 4.31. NO. PD is "*unimaginably*" stronger than IC, where "unimaginably" does not (yet) have technical meaning.

Fact 4.32 (Martin–Steel thm, 1985)**.** If there are n Woodin cardinals and a measurable cardinal above them, then $\mathsf{Det}(\boldsymbol{\Pi}^1_{n+1})$ holds.

This is tought of as the most remarkable thm in logic in 1980's, although we do not know what it says. The thm is named after Tony Martin and John Steel, and Woodin cardinals are named after Hugh Woodin.

Our next goal is to prove the case where $n = 0$.

§4.3 Measurable Cardinals

Definition 4.33. Fix a set S. $F \subseteq \mathcal{P}(S)$ is a **filter on** S, if

(1) If $X, Y \in F$, then $X \cap Y \in F$.

(2) If $X \in F$ and $Y \supseteq X$, then $Y \in F$.

(3) $\varnothing \notin F$.

(4) $S \in F$.

Think of $\in F$ as a notion of largeness.

A filter F is called **principal** if there is $s \in S$ s.t. $\{s\} \in F$. These look like $F = \{X : s \in X\}$, so we care about non-principal filters.

A filter F is **ultra** if $\forall X \subseteq S$ either $X \in F$ or $S \setminus X \in F$. So ultrafilters are maximal. Principal filters are ultra.

Let κ be a cardinal. We say F is **κ-complete** if

$$\forall \lambda < \kappa,\ \forall \{X_\alpha : \alpha < \lambda\} \subseteq F,\ \bigcap_{\alpha < \lambda} X_\alpha \in F.$$

So e.g. $\aleph_0$-completeness means closure under finite intersection and follows from (1). $\aleph_1$-completeness means ctbly complete or **σ-complete**. Principal filters are κ-complete $\forall \kappa$.

Question 4.34. When is there a κ-complete non-principal uf on κ?

We shall show that this implies κ is strongly inacc, but it is much stronger than that.

Although we now know PD unimaginably stronger than IC, people used 40 years due to lack of techniques.

Observe:

Proposition 4.35

If U is a κ-complete non-principal uf on κ, then $\forall \alpha < \kappa,\ \alpha \notin U$.

Proof. $|\alpha| < \kappa$, so α is a union of $< \kappa$ singletons, so $\alpha \notin U$ by κ-completeness. □

This also says $\kappa \setminus \alpha \in U$.

Corollary 4.36

No non-principal uf on κ can be λ-complete for $\lambda > \kappa$.

Proof.

$$\kappa = \bigcup_{\alpha \in \kappa} \{\alpha\}.$$

□

Proposition 4.37

If κ is unctble cardinal, and U is κ-complete uf, non-pr on κ, then κ is strongly inaccessible.

Remark 4.38. $\kappa = \aleph_0$ satisfies this property, so the study of inacc cardinals starts from "*wouldn't it be strange if there is only one such thing in the universe?*" Similar to the study of exoplanets / extraterrestrial life.

Proof. Regular Suppose κ is singular, then $\kappa = \bigcup_{\alpha<\lambda} \kappa_\alpha$ for $\kappa_\alpha < \kappa$. By Proposition 4.35, $\kappa_\alpha \notin U$, so by κ-completeness, $\bigcup_{\alpha<\lambda} \kappa_\alpha = \kappa \notin U$, contradiction.

Strong limit Suppose $\lambda < \kappa$ has $2^\lambda \geq \kappa$. Find $X \subseteq \{f : \lambda \to 2\}$ of cardinality κ. Let U^* be the uf on X inherited from U via the bijection $X \leftrightarrow \kappa$. Then U^* is non-pr and κ-complete.

We shall recursively construct some $y \in 2^\lambda$. Let $A_0 = X$. We can write $X = B \cup C$, where

$$B = \{x \in X : x(0) = 0\},\ C = \{x \in X : x(0) = 1\},$$

so exactly one of B or C is in U^*. Def $y(0) = \begin{cases} 0, & \text{if } B \in U^*, \\ 1, & \text{if } C \in U^*. \end{cases}$ Let $A_1 = \begin{cases} B, & \text{if } B \in U^*, \\ C, & \text{if } C \in U^*. \end{cases}$

For successors, suppose $y \restriction \alpha$ and A_α are defined, then let

$$B = \{x \in A_\alpha : (y \restriction \alpha)0 \subseteq x\},\ C = \{x \in A_\alpha : (y \restriction \alpha)1 \subseteq x\}.$$

By ind hyp $A_\alpha \in U^*$, so exactly one of B and C is in U^*, so we def $A_{\alpha+1}$ to be that one, and $y(\alpha)$ correspondingly.

For limits, def $A_\gamma = \bigcap_{\alpha<\gamma} A_\alpha$.

Claim 4.39. If γ limit, and $A_\alpha \in U^*\ \forall \alpha < \gamma$, then $A_\gamma \in U^*$.

Proof. This is precisely κ-completeness. □

Now we have defined $y \in 2^\lambda$ and A_α for $\alpha < \lambda$.

$$A = \bigcap_{\alpha<\lambda} A_\alpha \in U^*$$

by κ-completeness.

Claim 4.40. If $x \in A$, then $x = y$, i.e. $A \subseteq \{y\}$, either contradicting non-principality or that $\varnothing \notin U^*$.

Proof. If $\alpha < \lambda$, then $x(\alpha) = y(\alpha)$ by construction since $x \in A_{\alpha+1}$, and $A_{\alpha+1}$ is a set of those z s.t. $z(\alpha) = y(\alpha)$. □

□

Definition 4.41. A uf U on κ is called **normal** if for any family $\{A_\alpha : \alpha < \kappa\} \subseteq U$ (note that this has length κ unlike that for κ-completeness), the diagonal intersection

$$\triangle_{\alpha<\kappa} A_\alpha = \left\{ \xi < \kappa : \xi \in \bigcap_{\alpha<\xi} A_\alpha \right\}$$

is in U.

Note that ξ appears in the subscript of $\bigcap$.

Definition 4.42. κ cardinal is called **measurable** if there is a κ-complete non-pr normal uf on κ.

Theorem 4.43

ZF$\underline{\text{C}}$ $\models$ If there is a κ-complete non-pr uf on κ, then we can normalise the uf. This says there is a κ-complete non-pr normal uf on κ, i.e. κ is measurable.

Remark 4.44. We shall not prove this. Highly-non-trivially uses AC. There is a "canonical candidate" for the normalisation function.

GOAL:

Theorem 4.45 (Martin's thm)

If there is a measurable cardinal, then all $\boldsymbol{\Pi}^1_1$-sets are determined.

This is the $n = 0$ case of Martin–Steel.

IDEA: Tree rep. If T is a tree on any M, then in ZFC, $G([T])$ is determined by Gale–Stewart (if T is well-ordered then we do not need AC, e.g. when $M = \kappa$ or $\kappa \times \omega^n$). If $A \in \boldsymbol{\Sigma}^1_1$, then $A = p[T]$ where $T \subseteq (\omega \times \omega)^{<\omega}$ is well-orded, so $G([T])$ is determined. For the game $G(A)$:

$$G(A) \quad \begin{array}{l|ccccccc} \text{I} & x_0 & & x_2 & & \cdots & & x, \\ \text{II} & & x_1 & & x_3 & & & \end{array}$$

consider the **auxilliary game** $G_{\text{aux}}(A, T)$:

$$G_{\text{aux}}(A,T) \quad \begin{array}{l|cccccccc} \text{I} & y_0, x_0 & & y_1, x_2 & & y_2, x_4 & \cdots & & (y,x) \\ \text{II} & & x_1 & & x_3 & & & & \end{array}$$

where I wins if $(y, x) \in [T]$.

By above, $G_{\text{aux}}(A, T)$ is determined.

1. If I has a win strat in $G_{\text{aux}}(A, T)$, then I can extract x_{2n} from the win strat in $G_{\text{aux}}(A, T)$ to obtain a win strat in $G(A)$ (i.e. I plays G_{aux} in I's mind while II thinks II is playing G). The play in $G_{\text{aux}}(A, T)$ is $(y, x) \in T$ which witnesses $x \in p[T] = A$.

   We are getting close to the highly non-trivial part of the course.

2. In order to get a win strat for II for $G(A)$ from $G_{\text{aux}}(T)$, we need a translation function.

Definition 4.46. A set $A \subseteq \omega^\omega$ is ***κ*-Suslin** if there is a tree $T \subseteq (\kappa \times \omega)^{<\omega}$ s.t. $A = p[T]$. (See Sheet 3 Problem 40)

Can show $\aleph_0$-Suslin $\iff$ $\boldsymbol{\Sigma}^1_1$. This means if we have tree rep for $\boldsymbol{\Pi}^1_1$, then $\kappa > \aleph_0$.

Theorem 4.47 (Shoenfield)

$\boldsymbol{\Pi}^1_1$ sets are $\aleph_1$-Suslin.

Proof. Let A be Π^1_1. Let T be the tree rep of $\omega^\omega \setminus A$. Then $T \subseteq (\omega \times \omega)^{<\omega}$, and $x \notin A \iff x \in p[T]$. Let

$$T_x = \{s \in \omega^{<\omega} : (s, x \restriction |s|) \in T\}.$$

Then

$$\begin{aligned} x \in A &\iff T_x \text{ is wellfdd} && \text{previous result} \\ &\iff \exists \text{ order-preserving map } (T_x, \supseteq) \to (\alpha, \leq) && \text{by Problem 30.} \end{aligned}$$

By ctbility, $\alpha = \aleph_1$ is enough.

We want $\widehat{T} \subseteq (\aleph_1 \times \omega)^{<\omega}$ s.t. $\exists g,\ (g, x) \in [\widehat{T}] \iff x \in A$. Note that g is a function $\omega \to \aleph_1$.

Encoding order-preserving maps Fix a list of all finite sequence of natural numbers $i \mapsto s_i$ bijective. We way $g : \omega \to \aleph_1$ is an **order-preserving code** (OPC) for a tree S if $\forall i, j$ with $s_i, s_j \in S$,

$$s_i \supseteq s_j \implies g(i) \leq g(j). \tag{$*$}$$

This is equiv to saying $f : S \to \aleph_1,\ s_i \mapsto g(i)$ is ord-preserving.

We say $u : n \to \aleph_1$ is a partial opc for S if $(*)$ holds on $\operatorname{dom} u$.

First attempt Let

$$\widehat{T} = \{(u, s) : s \in \omega^{<\omega}, u \text{ is a partial opc for } T_x\}.$$

This does not work because there is an x. There are a lot of choices of extensions x given s, but we only know s.

$$T_x = \bigcup_{n \in \omega} T_{x \restriction n}, \text{ where } T_{x \restriction n} = \{s : |s| \leq n, \text{ and } (s, x \restriction |s|) \in T\}$$

is a finite height tree.

Second attempt

$$\widehat{T} = \{(u, s) : s \in \omega^{<\omega} \text{ and } u \text{ is a partial opc for } T_s\}.$$

This is called the **Shoenfield tree**.

Claim 4.48. $A = p[\widehat{T}]$.

Proof.

$$\begin{aligned} x \in A &\iff \exists f : (T_x, \supseteq) \to (\aleph_1, \leq) \text{ order-preserving} \\ &\iff \exists g \in \aleph_1^\omega, g \text{ is an opc for } T_x \\ &\overset{?}{\implies} (g, x) \in [\widehat{T}]. \end{aligned}$$

To see (?), fix n, and consider $g \restriction n$ and $x \restriction n$.

$$\begin{aligned} g : n \to \aleph_1 \text{ is a partial opc for } T_x &\implies g \restriction n \text{ is a partial opc for } T_{x \restriction n} \\ &\implies (g \restriction n, x \restriction n) \in \widehat{T}. \end{aligned}$$

Conversely,

$$\begin{aligned} x \in p[\widehat{T}] &\implies \exists g : \omega \to \aleph_1 \text{ s.t. } (g, x) \in [\widehat{T}] \\ &\implies \exists g \forall n,\ g \restriction n \text{ is partial opc for } T_{x \restriction n}. \end{aligned}$$

Suppose i, j are s.t. $s_i, s_j \in T_x$. Find large n s.t. $s_i, s_j \in T_{x \upharpoonright n}$. Then $g \upharpoonright n$ partial opc says $g(i) \leq g(j)$, so g is opc for T_x, so

$$\begin{aligned} \exists g \forall n,\ g \upharpoonright n \text{ is partial opc for } T_{x \upharpoonright n} &\implies \exists g,\ g \text{ opc for } T_x \\ &\implies x \in A. \end{aligned}$$

□

□

Remark 4.49. The Shoenfield tree for A does not depend on A but only on T. This was used by Shoenfield to obtain **Shoenfield absoluteness**: if $M \supseteq \aleph_1^N$, then M and N agree on Π^1_1 sentences. Π^1_1 sentences are a lot, e.g. statements about naturals, and $\mathsf{Cons}(\mathsf{ZFC})$ is a statement about naturals, namely that $\forall n$, n does not encode a proof of $0 = 1$.

Proof of Theorem 4.45. Let κ be measurable. Observe that the Shoenfield tree construction works for any cardinal $\kappa \geq \aleph_1$. So let

$$\widehat{T} = \{(u, s) : u \in \kappa^{<\omega},\ s \in \omega^{<\omega},\ u \text{ is a partial opc for } T_s\}.$$

The aux game $G_{\text{aux}}(\widehat{T})$:

$$\begin{array}{llllllll} \text{I} & x_0, \alpha_0 & & x_2, \alpha_1 & & x_4, \alpha_2 & \cdots & x = (x_i : i \in \omega) \\ \text{II} & & x_1 & & x_3 & & & g(i) = \alpha_i, \end{array}$$

and (g, x) is a win for I if $(g, x) \in [\widehat{T}]$ is determined.

WTP that we can translate win strat τ for II in $G_{\text{aux}}(\widehat{T})$ into one τ^* for II in $G(A)$. We already have problem in the first step because we have to guess out of thin air α_0 corresponding to x_0, in order to play $\tau(x_0, \alpha_0)$. Also if we can do this in all steps then we are done.

Sheet 3 Problem 32 says there is the Kleene–Brouwer order $<_{\text{KB}}$ on $\omega^{<\omega}$ s.t. $(T, \supseteq)$ is wellfdd $\iff$ $(T, <_{\text{KB}})$ is a well ord, i.e. we can linearise tree orders.

A function $g : \omega \to \kappa$ is called a **KB-code** for S if

$$\forall i, j \text{ with } s_i, s_j \in S,\ s_i <_{\text{KB}} s_j \implies g(i) < g(j). \tag{$*$}$$

Simly $g : n \to \kappa$ is a **partial KB-code** for S if $(*)$ holds for $i, j < n$.

The KB-variant of Shoenfield tree is now

$$T^{\text{KB}} = \{(u, s) : u \text{ is a partial KB-code on } T_s\},$$

and $A = p[T^{\text{KB}}]$.

N.b. we have not used κ measurable but only slightly modified the tree. To summarise our progress, for $\mathbf{\Pi}^1_1$ set A,

$$\begin{aligned} x \in A &\iff x \in p[\widehat{T}] \\ &\iff x \in p[T^{\text{KB}}],\ T^{\text{KB}} \subseteq (\aleph_1 \times \omega)^{<\omega}. \end{aligned}$$

If I wins $G_{\text{aux}}(T^{\text{KB}})$, then I wins $G(A)$.

WTP: If II wins $G_{\text{aux}}(T^{\text{KB}})$, then II wins $G(A)$. $G_{\text{aux}}(T^{\text{KB}})$ is closed so must be determined by Gale–Stewart, so this shows $G(A)$ determined.

Let τ be a win strat for II in $G_{\text{aux}}(T^{\text{KB}})$. Suppose $s \in \omega^{<\omega}$ and $|T_s| = m$. Take $Q \subseteq \kappa$ of size m. Then we have an order isomorphism $(T_s, <^{\text{KB}}) \leftrightarrow (Q, <)$ sending i_n to q_n, where $T_s = \{s_{i_0}, s_{i_1}, \cdots\}$ and $Q = \{q_0, q_1, \cdots\}$.

Let

$$u_Q(i) = \begin{cases} \text{the } j\text{-th element of } Q, & \text{if } s_i \text{ is } s_{i_j}, \\ 0, & \text{if } s_i \notin T_s. \end{cases}$$

Then u_Q is a partial KB-code.

Play the game $G_{\text{aux}}(T^{\text{KB}})$ by filling the missing I-moves with appropriate elements of $Q \cup \{0\}$, that is, fix Q, and II interprets the position as

$$\begin{array}{lcccccc} \text{I} & x_0, u_Q(0) & & x_2, u_Q(1) & & x_4, u_Q(2) & \cdots \\ \text{II} & & x_1 & & x_3 & & \end{array}$$

Suppose given $(x_0, x_1, \cdots, x_{2n}) = s$, then define $f_s(Q) = \tau(u_Q \restriction n, s)$.

f_s is a colouring of $\kappa^{(m)}$ with $\aleph_0$ many colours. By Rowbottom's thm (Problem 39), there is a monochromatic $H_s \in U$ (the ultrafilter witnessing κ measurable). Since U is κ-complete, and there are only $\aleph_0$ many sequences s, we get that

$$H = \bigcap_{s \in \omega^{<\omega}} H_s \in U,$$

so H is unctble, so if $x \in A$, then there is an order-preserving injection $(T_x, <_{\text{KB}}) \to (H, <)$. Now we shall use H to define a strat τ_H in the game $G(A)$. Say the current position is

$$\begin{array}{lccccc} \text{I} & x_0 & & x_2 & \cdots & x_{2n} \\ \text{II} & & x_1 & & x_3 & \end{array}$$

Let $s = (x_0 x_1 \cdots x_{2n})$. If $m = |T_s|$, then $\forall Q, Q' \in H^{(m)},\ f_s(Q) = f_s(Q')$, so we can def $\tau(s)$ to be the unique value of f_s on elements of $H^{(m)}$.

Claim 4.50. τ_H is a win strat for II in $G(A)$.

Suppose not, then let $x \in A$ be the result of playing by τ. Since $x \in A$, there is an order-preserving injection $h : (T_x, <_{\text{KB}}) \to (H, <)$. Write

$$g(i) = \begin{cases} \alpha, & \text{if } s_i \in T_x \text{ and } h(s_i) = \alpha, \\ 0, & \text{if } s_i \notin T_x. \end{cases}$$

Then g is a KB-code for T_x, so in particular, $(g, x) \in [T^{\text{KB}}]$.

We shall show that (g, x) is a play in $G_{\text{aux}}(T^{\text{KB}})$ following a win strat τ for II. Consider the following position in $G_{\text{aux}}(T^{\text{KB}})$:

$$\begin{array}{lccccc} \text{I} & x_0, u(0) & & x_2, u(1) & \cdots & \\ \text{II} & & x_1 & & & x_{2n+1} \end{array}$$

Here

$$\tau(u \restriction n, \underbrace{(x_0 \cdots x_{2n})}_{s}) = f_s(Q) = \tau_H(s) = x_{2n+1},$$

Q arbitrary in $H^{(m)}$, so (g, x) is according to τ in $G_{\text{aux}}(T^{\text{KB}})$, so $(g, x) \notin [T^{\text{KB}}]$, contradiction. $\square$

Remark 4.51. We have now reached the 1970's, when people prove that determinacy follows from large cardinals.

§5 Axiom of Determinacy

Definition 5.1 ($\mathsf{AD}, \mathsf{AD}_X$)**.** AD says "For all $A \subseteq \omega^\omega$, $G(A)$ is determined." We also have the variant AD_X which says "For all $A \subseteq X^\omega$, $G(A)$ is determined."

Remark 5.2. We have seen that $\mathsf{AD} \implies \neg\mathsf{AC}$. Historically this is proposed as an alternative to AC that attracted attack (afterall, AC is TRUE (?)).

We have already seen

1. If AD holds, then every $A \subseteq \omega^\omega$ has PSP.
2. Boundedness thm for WO. (Did this use CHOICE?)

One thing that uses AC but few people know uses AC:

$$f : \omega^\omega \to \omega^\omega \text{ is cts} \iff \forall \text{ seq } (x_n) \to x,\ (f(x_n)) \to f(x) \quad \text{"sequential continuity"}.$$

This uses the general equivalence in metric spaces, whose proof uses a small fraction of AC, namely $\mathsf{AC}_\omega(\mathbb{R})$, i.e. picking from a ctble family of subsets of $\mathbb{R}$, when we try to find a sequence $x_n \to x$ with $f(x_n) \not\to f(x)$ given f not cts.

Theorem 5.3

$\mathsf{AD} \implies \mathsf{AC}_\omega(\mathbb{R})$. More generally, $\mathsf{AD}_X \implies \mathsf{AC}_X(X^\omega)$.

Proof. In Sheet 1, we have shown that finite determinacy for games on X gives $\mathsf{AC}_X(X)$. Here it is $\mathsf{AC}_\omega(\omega)$ which is obvious by well-orderedness.

But now we have inf games

$$\begin{array}{lll} \text{I} & x_0 & \\ \text{II} & & x_1 x_2 \cdots \end{array}$$

with II wins if $(x_1 x_2 \cdots) \in A_{x_0}$. As on Sheet 1, I cannot have win strat, so by AD, II has, so the win strat for II is a choice function for $\{A_x : x \in \omega\}$, so $\mathsf{AD}_X \implies \mathsf{AC}_X(X^\omega)$. □

Question 5.4. Can we use any of the descriptive set theory which is developed under ZFC?

Almost all of it can be done in $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$, but we really need this.

Recall the Feferman–Lévy model M of ZF where $\mathbb{R}$ is a ctble union of ctble sets, but $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R}) \vdash$"ctble union of ctble real sets is ctble", and $\mathsf{ZF} \vdash \mathbb{R}$ is unctble, so $M \models \neg\mathsf{AC}_\omega(\mathbb{R})$. In M, every set of reals is $\boldsymbol{\Delta}^0_4$.

Proposition 5.5

$\mathsf{ZF} \not\vdash$ Borel-Determinacy.

Proof. If it does, then $M \models$ Borel-Determinacy, so $M \models \mathsf{AD}$. By previous arguments, the constructible universe $\boldsymbol{L}$ has a copy $\boldsymbol{L}^M$ in M, and $\boldsymbol{L}^M \models \mathsf{ZFC} + \mathsf{IC}$, but this contradicts Gödel's incompleteness.

Alternatively, $M \models \mathsf{AD}$, so $M \models \mathsf{AC}_\omega(\mathbb{R})$, contradicting $M \models \neg\mathsf{AC}_\omega(\mathbb{R})$. □

Fact 5.6. If $\iota : X \hookrightarrow Y$ is injective, then $\mathsf{AC}_Z(Y) \implies \mathsf{AC}_Z(X)$ and $\mathsf{AD}_Y \implies \mathsf{AD}_X$.

Corollary 5.7

$\mathsf{AD}_{\aleph_1}$ is inconsistent (with ZF).

Proof.

1. $\mathsf{AD}_{\aleph_1} \implies \mathsf{AD} \implies$ every set has PSP.
2. We know that unctble sets of unique codes cannot have PSP.
3. A set of unique codes can be produced by a choice function for the family $\{\mathrm{WO}_\alpha : \alpha < \aleph_1\}$
4. This is a consequence of $\mathsf{AC}_{\aleph_1}(\omega^\omega)$.
5. But $\mathsf{AD}_{\aleph_1} \implies \mathsf{AC}_{\aleph_1}(\aleph_1^\omega) \implies \mathsf{AC}_{\aleph_1}(\omega^\omega)$.

$\square$

Remark 5.8. It is curious that different models of ZF can have very different witnesses to $\neg\mathsf{AD}_{\aleph_1}$. It is either a non-determined game of the form $G^*(A)$, $A \subseteq \omega^\omega$, or the choice function game as in $\mathsf{AC}_{\aleph_1}(\aleph_1^\omega)$ proof.

We have seen so far

$$\begin{tikzcd}
\mathsf{AD} \arrow[r, rightsquigarrow] & \mathsf{PD} \arrow[r, rightsquigarrow] \arrow[dr, rightsquigarrow] & \mathsf{Det}(\mathbf{\Pi}^1_1) \\
 & \mathsf{MC} \arrow[r, rightsquigarrow] \arrow[ur, rightsquigarrow] & \mathsf{IC}
\end{tikzcd}$$

where $\rightsquigarrow$ means "is stronger than", PD is proj determinacy, MC is measurable cardinal, and IC is inacc cardinal.

Our goal for the final three lectures is to prove that AD is stronger than MC. In fact the following pairs are roughly equiv in strength (whatever that means):

$$\begin{aligned}
\mathsf{Det}(\mathbf{\Pi}^1_1) &\leftrightsquigarrow \mathsf{MC}, \\
\mathsf{PD} &\leftrightsquigarrow \forall n, \exists n \text{ Woodin Cardinals}, \\
\mathsf{AD} &\leftrightsquigarrow \exists \infty \text{ Woodin Cardinals}.
\end{aligned}$$

See Kanamori, *The Higher Infinite*, Chapters 31 and 32. "*This is a book that you can read late at night (if you skip the proofs)*"

Theorem 5.9 (Solovay)

$\mathsf{AD} \implies$ There is a non-pr $\aleph_1$-complete uf on $\aleph_1$.

Note that such a cardinal is strongly inacc in ZFC, but only the "regular" part of our proof works without AC. First step is to show $\mathsf{AD} \implies$ every uf is $\aleph_1$-complete.

At this stage it is conventional to state the axioms assumed in brackets next to the theorem number.

Lemma 5.10 (ZF)

If U is a uf on S and U is not $\aleph_1$-complete, then there is a non-pr uf on $\mathbb{N}$.

Proof. $S = \bigcup_{n\in\omega} A_n$ with $A_n \notin U$, $A_i \cap A_j \neq \varnothing$. Def $f : S \to \mathbb{N}$ by $x \mapsto n$ if $x \in A_n$. Then consider $U^* = f_*U = \{X \subseteq \mathbb{N} : f^{-1}[X] \in U\}$. By checking the definitions, U^* is an uf on $\mathbb{N}$.

WTP U^* is non-pr. Assume $\{n\} \in U^*$, then $f^{-1}[\{n\}] \in U$, i.e. $A_n \in U$, contradiction. □

Proposition 5.11 (ZF + AD)
There is no non-pr uf on $\mathbb{N}$.

Proof. Let U be any uf on $\mathbb{N}$. Def G_U:

I	s_0		s_2	$\cdots$
II		s_1		

where $s_i \in \mathbb{N}^{(<\omega)}$ are finite subsets, and

$$s_i \cap \bigcup_{j\neq i} s_j = \varnothing. \tag{$*$}$$

If $(*)$ is violated, then the first player who does loses. Otherwise, let $A = \bigcup_{i\in\omega} s_{2i}$, $B = \bigcup_{i\in\omega} s_{2i+1}$, then A, B are disjoint, so at most one is in U. I wins if $A \in U$. If U is principal, then I has win strat. We shall show that if U is non-pr then no one has win strat by **strategy stealing**.

1. Suppose σ is a win strat for I, then we need to construct τ for II winning against sequence $s_0, s_2, \cdots$ played by I.

	I	s_0		s_2		s_4	$\cdots$	A
	II		$s_1 = t_1 \setminus s_0$		$s_3 = t_2 \setminus s_0$		$\cdots$	B
G_{aux}	I'(= II)	$t_0 = \sigma(\varnothing)$		$t_2 = \sigma(t_0t_1)$			$\cdots$	A'
	II'(= I)		$t_1 = s_2$		$t_3 = s_4$		$\cdots$	B'

   We define τ by $\sigma(t_0, \cdots, t_{2i-1}) \setminus s_0$ as above. Call the unions of each row A, B, A', B'. Since σ is winning, $A' \in U$, and $A = B' \cup s_0$. Since $A' \in U$, $B' \notin U$, so $A \in U \iff s_0 \in U$, so $A \in U \iff U$ is principal.

   Hence if U is non-pr, then τ is a win strat for II, thus I cannot have win strat.

2. Suppose τ is a win strat for II. Fix $s_1, \cdots$ moves for II. We define a strat for I as follows:

	I	$s_0 = t_1$		$s_2 = t_3$		$\cdots$	A
	II		s_1		s_3	$\cdots$	B
G_{aux}	I'(= II)	$t_0 = \varnothing$		$t_2 = s_1$		$\cdots$	A'
	II'(= I)		$t_1 = \tau(t_0)$		$t_3 = \tau(t_0t_1t_2)$	$\cdots$	B'

   Since τ was winning for II, $A' \notin U$, so $B = A' \notin U$. However, this is not enough for $A \in U$ since we do not know $A = \omega \setminus B$. But this is just technicality.

   <u>Tech aside:</u> If τ is any strat for II, then we can replace it by τ' that guarantees $B = \omega \setminus A$ and s.t. if τ is win for II then τ' is also win for II.

   τ' is defined as: do what τ does but in addition, check at round k which numbers $\leq k$ are still open for grab. So we could have done this first and WLOG $A = \omega \setminus B$.

Now $A = \omega \setminus B \in U$, so σ is winning. Thus every win strat for II can be transformed to a win strat for I, so II has no win strat.

□

Remark 5.12. Zorn's lemma gives 2^{2^κ} many ufs on κ, but there are only κ many pr ufs! So most of them are non-pr.

But AD says the number of ufs on ω is $\aleph_0 < 2^{2^{\aleph_0}}$. As a consequence, the hard part is not completeness (given AD this is automatic up to $\aleph_1$), but non-principality (easy by AC but difficult for AD).

We shall do Martin's construction of an uf on $\aleph_1$. Consider $\mathcal{R} = (V_{\omega+1}, \in)$. What is in $\mathcal{R}$?

If $x \in \omega^\omega$, then elements of x are of the form

$$(n, m) = \{\{n\}, \{n, m\}\} \in V_\omega,$$

so $x \subseteq V_\omega$, $x \in V_{\omega+1}$, so $\omega^\omega \subseteq V_{\omega+1}$.

If σ is a strategy $\omega^{<\omega} \to \omega$, $s \in \omega^{<\omega}$, then $s \in V_\omega$, so $\sigma \subseteq V_\omega$, so $\sigma \in V_{\omega+1}$.

We call a formula φ in two vars **$\mathcal{R}$-absolute** if for all transitive models $M \subseteq V$ of ZF and $\forall x, y \in M$,

$$M \models \text{“}\mathcal{R} \models \varphi(x, y)\text{”} \iff \mathcal{R} \models \varphi(x, y).$$

Here M is a ZF model so it thinks some subset of M as $V_{\omega+1}^M$ which may not be the true $V_{\omega+1}$.

Example 5.13 ($\mathcal{R}$-absolute formulae)

1. Quantifier-free formulae.
2. Formulae where all quantifiers are of the form $\exists u \in \mathbb{N}$. (Since transitive models agree on what $\mathbb{N}$ is.)
3. Others that Kunen's book in section on absoluteness details.

Definition 5.14 ($\mathcal{R}$-absolutely definable)**.** For $x, y \in V_{\omega+1}$, if there is φ $\mathcal{R}$-absolute s.t.

$$\forall z, \quad z \in x \iff (V_{\omega+1}, \in) \models \varphi(z, y),$$

then we say x is **$\mathcal{R}$-absolutely definable** in y, written $x \leq_{\mathrm{D}} y$.

$\leq_{\mathrm{D}}$ is reflexive and transitive (conjunctions are absolute), so $\leq_{\mathrm{D}}$ is a **partial preorder**, but not in general antisymmetric: $\varnothing \leq_{\mathrm{D}} \{\varnothing\}$, $\{\varnothing\} \leq_{\mathrm{D}} \varnothing$.

For partial preorders, we can define $x \equiv_{\mathrm{D}} y$ if $x \leq_{\mathrm{D}} y$ and $y \leq_{\mathrm{D}} x$. Then $\equiv_{\mathrm{D}}$ is an equiv relation, and quotienting gives a partial order.

Since $x \leq_{\mathrm{D}} y$ is witnessed by some φ and there are only ctbly many φ, we have $\forall y$, $\{x : x \leq_{\mathrm{D}} y\}$ is ctble. So each equiv class $[y]_{\equiv_{\mathrm{D}}}$ is ctble.

Since there is an $\mathcal{R}$-absolute def of a bijection between $\omega^{<\omega}$ and ω, we can encode strats σ as elements of ω^ω, say $\mathrm{code}(\sigma) \in \omega^\omega$ s.t. $\mathrm{code}(\sigma) \equiv_{\mathrm{D}} \sigma$. If $x, y, \in \omega^\omega$, def

$$x \oplus y(i) = \begin{cases} x(k), & \text{if } i = 2k, \\ y(k), & \text{if } i = 2k+1, \end{cases}$$

then $x \leq_{\mathrm{D}} x \oplus y$, and $y \leq_{\mathrm{D}} x \oplus y$, so $x_{\mathrm{I}} \leq_{\mathrm{D}} x$ and $x_{\mathrm{II}} \leq_{\mathrm{D}} x$.

Therefore $\sigma * x \leq_{\mathrm{D}} \mathrm{code}(\sigma) \oplus x$. This $\oplus$ is called **Turing sum**. We shall def the Martin filter on ω^ω using $\leq_{\mathrm{D}}$ with the prop that

$$\mathsf{AD} \implies \text{the Martin filter on } \equiv_{\mathrm{D}} \text{-closed sets is a non-pr uf.}$$

But we will need to map ω^ω to $\aleph_1$ that preserves non-principality of the uf.

Definition 5.15. A set $C \subseteq \omega^\omega$ is called a **cone** (**D-cone**) if there is $x \in \omega^\omega$ s.t. $C = \{y \in \omega^\omega : x \leq_{\mathrm{D}} y\} = \mathrm{cone}(x)$. x is called the **basis** of the cone.

Definition 5.16 (Martin)**.**

$$M_{\mathrm{D}} = \{A \subseteq \omega^\omega : A \text{ includes a cone}\}$$

is the **Martin filter**.

Proposition 5.17 (ZF)
M_{D} is a non-pr filter on ω^ω.

Proof.

(1) If $A \in M_{\mathrm{D}}$, $B \supseteq A$, then B includes a cone.

(2) $\varnothing$ does not include a cone, and ω^ω includes (is) a cone.

(3) Let $C \subseteq A$, $D \subseteq B$ be cones, then let $C = \mathrm{cone}(x)$, $D = \mathrm{cone}(y)$. Consider $E = \mathrm{cone}(x \oplus y)$. Since $x \leq_{\mathrm{D}} x \oplus y$ and $y \leq_{\mathrm{D}} x \oplus y$, $E \subseteq C \cap D$, so $E \subseteq A \cap B$, so $A \cap B \in M_{\mathrm{D}}$.

(4) If $A \subseteq \omega^\omega$ is finite, then A cannot include a cone, so M_{D} is non-pr.

□

Remark 5.18. In ZFC, it is also true that M_{D} is $\aleph_1$-complete because in (3), we let $A_n \in M_{\mathrm{D}}$ and $C_n = \mathrm{cone}(x_n) \subseteq A_n$ by AC, then construct $\bigoplus_{n\in\omega} x_n$ as the unique y s.t. $(y)_n = x_n$ with an explicit bijection $\omega \times \omega \to \omega$.

This argument only works in ZFC, so in our later ZF + AD application, we need to use our previous thm about $\aleph_1$-completeness.

Lemma 5.19 (Martin's)
Suppose $A \subseteq \omega^\omega$ s.t. A is closed under $\equiv_{\mathrm{D}}$, i.e. if $x \in A$, and $y \equiv_{\mathrm{D}} x$, then $y \in A$. If I has w.s. in $G(A)$, then A includes a cone. If II has w.s. in $G(A)$, then $\omega^\omega \setminus A$ includes a cone.

Corollary 5.20
$\mathsf{AD} \models M_{\mathrm{D}} \cap \{A : A \equiv_{\mathrm{D}} \text{-closed}\}$ is an $\aleph_1$-complete non-pr uf on $\{A : A \equiv_{\mathrm{D}} \text{-closed}\}$, which is equiv to the same on the quotient $\mathcal{D}_{\mathrm{D}} = \omega^\omega / \equiv_{\mathrm{D}}$.

Proof of Lemma 5.19. (For I only). Let σ be a w.s. for I.

Claim 5.21. $C = \text{cone}(\text{code}(\sigma)) \subseteq A$.

Proof. Let $z \in C$, i.e. $z \geq_{\mathrm{D}} \sigma \equiv_{\mathrm{D}} \text{code}(\sigma)$. Then

$$z \leq_{\mathrm{D}} \sigma * z \leq_{\mathrm{D}} \text{code}(\sigma) \oplus z \leq_{\mathrm{D}} z.$$

Hence $\sigma * z \equiv_{\mathrm{D}} z$, and $\sigma * z \in A$, so $z \in A$. □

□

Fact 5.22. Martin was a recursion theorist before proving BorelD which made him become set theorist. This lem was his first contribution to set theory but he did not think of it as a set theory thing.

Now we have a $\aleph_1$-complete non-pr uf on $\mathcal{D}_{\mathrm{D}}$. We need to transfer it to $\aleph_1$.

Recall **Gödel's real model family $\boldsymbol{L}_x$**. These are transitive models of set theory s.t. $x \in \boldsymbol{L}_x$, with the properties that each is projly well-orded, and $\boldsymbol{L}_x \models \mathsf{CH}$. We used this to show that if every proj set has the PSP, then $\aleph_1$ is inacc by reals for $(\boldsymbol{L}_x : x \in \omega^\omega)$. So $\forall x, \aleph_1^{\boldsymbol{L}_x} < \aleph_1$.

Fact 5.23. If $x \equiv_{\mathrm{D}} y$, then $\boldsymbol{L}_x = \boldsymbol{L}_y$. So the usual notation is $\boldsymbol{L}(x)$ for $\boldsymbol{L}_x$ as there is a uniform def for $\boldsymbol{L}(x)$ given x.

In $\mathsf{ZF} + \mathsf{AC}$, every set has PSP, so $\aleph_1$ is inacc by reals for $(\boldsymbol{L}(x) : x \in \omega^\omega)$.

Remark 5.24. $\aleph_1$ is inacc by reals for $(\boldsymbol{L}(x) : x \in \omega^\omega)$ talks about more than $\aleph_1$.

Claim 5.25. $\forall x,\ \aleph_n^{\boldsymbol{L}(x)} < \aleph_1$.

Proof. $\aleph_1^{\boldsymbol{L}(x)} < \aleph_1$, so let $y \in \mathrm{WO}$ s.t. $\|y\| = \aleph_1^{\boldsymbol{L}(x)}$. In $\boldsymbol{L}(x \oplus y)$, we have $\aleph_1^{\boldsymbol{L}(x \oplus y)} < \aleph_1$, so $\aleph_2^{\boldsymbol{L}(x)} \leq \aleph_1^{\boldsymbol{L}(x \oplus y)} < \aleph_1$(?. Simly $\aleph_n^{\boldsymbol{L}(x)} < \aleph_1$. □

Let $a : \omega^\omega / \equiv_{\mathrm{D}} \to \aleph_1$ by $x \mapsto \aleph_1^{\boldsymbol{L}(x)}$. This is well-def by Fact 5.23. Def U on $\aleph_1$ by $A \in U \iff a^{-1}[A] \in M_{\mathrm{D}}$ (the **pushforward** of the Martin filter). U is then an $\aleph_1$-complete uf on $\aleph_1$. WTP non-principality.

Suppose $\{\alpha\} \in U$. Then $a^{-1}[\{\alpha\}] \in M_{\mathrm{D}}$, so there is a cone $C \subseteq a^{-1}[\{\alpha\}]$. Let $C = \text{cone}(x)$. Then $\forall z$, if $x \leq_{\mathrm{D}} z$, then $\aleph_1^{\boldsymbol{L}(z)} = \alpha$.

Let y be s.t. $\|y\| = \aleph_1^{\boldsymbol{L}(z)}$ and consider $z \oplus y$. We have $x \leq_{\mathrm{D}} z \leq_{\mathrm{D}} z \oplus y$, but $\aleph_1^{\boldsymbol{L}(z \oplus y)} > \aleph_1^{\boldsymbol{L}(z)}$, so no cone is included by the preimage of a singleton.

The last and non-examinable lecture on further consequences of AD did not happen, but notes from the lecturer is available at Lent2020/infinitegames_lecture24.pdf on the lecturer's page. Archived from the original on 22 Mar 2020.